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ABSTRACT

Sixteen mathematicians from universities, schools and industries met for five days in 1966 to plan a "second round" curriculum which would place greater emphasis on the relevance of mathematics to problems of the real world. Their deliberations are reported in this document. The three main recommendations were that the curriculum should contain: (1) frequent consideration of mathematical models of significant problem situations; (2) an introduction to those mathematical concepts which are important to the general citizen; and (3) probability, logic, computing and flow charts, and the concept of function, in addition to traditional topics. A series of reports and papers on various topics contain many suggestions for putting these recommendations into effect, especially in grades seven through nine. Also included are an annotated bibliography on applications of mathematics, and a list of topics for the use of mathematical models. (MM)

SCHOOL MATHEMATICS
STUDY GROUP

REPORT OF A CONFERENCE ON SECONDARY SCHOOL MATHEMATICS
NEW ORLEANS MARCH 14-18, 1966

SMSG Working Paper

Not for outside distribution

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March Planning Session - Secondary School Mathematics
March 14-18, 1966

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PREFACE

A conference was held March 14-18, 1966, in New Orleans to discuss plans for a second consideration by the School Mathematics Study Group of the secondary school mathematics curriculum.

The participants in the conference were asked to discuss the mathematics which might be included in a new program for grades seven through twelve and to prepare recommendations for consideration by those who will prepare detailed outlines of a new curriculum.

After preliminary discussions, the conference agreed on two basic recommendations. First, the conference applauded the desire already expressed by the SMSG Advisory Board to attempt to make clear to students the relevance of mathematics to problems of the real world and recommended that this be done by frequent consideration of mathematical models of significant and interesting problem situations in a variety of areas.

Secondly, the conference recommended that the curriculum for grades seven through nine contain an introduction to those mathematical concepts which are important to the general citizen. It was intended that this curriculum would be for all students, with the less able moving at a slower pace and taking longer to complete it.

In addition to the usual topics in arithmetic, geometry, and algebra usually studied in grades seven through nine, the conference suggested that consideration be given to probability, logic, computing and flow charts, and the concept of function.

The procedure followed during the conference was to meet in small committees of three or four to study individual topics. The reports of these committees were then discussed and revised by the entire conference. The process was then repeated with new committees and new topics, some of which were suggested by the discussion of previous reports.

At the conclusion of the conference, a first draft of this report was circulated to the participants. Such corrections as were received were used in preparing the final draft. In addition, Professor Buck submitted a number of supplementary suggestions, and these are appended to this report.

An annotated bibliography, prepared by Max Bell, of articles dealing with applications of mathematics and mathematical models is found at the end of the report.

Finally, it must be emphasized that the recommendations and suggestions of this report are not to be taken as official SMSG policy. While the SMSG Advisory Board has expressed general approval of the spirit of this report, the report itself is no more than is indicated in the second paragraph of this preface.

Report of the Committee on Mathematical Models

Aims

The activity of constructing and analyzing mathematical models of scientific and life situations is, apart from technical questions, the principal link between mathematics and the rest of civilization. How much training can be given in this activity in 7-9 (or 10-12, for that matter) is not at all obvious at this stage, but the importance of the activity is so clear that an attempt seems worthwhile. Specific skills to be inculcated and specific insights to be presented are as follows:

A. Skills

1. Drawing of diagrams, tables, flow charts, etc., to describe situations.

2. Familiarity with useful symbols and functions, such as max, min, $(\ )_+$, $|\ |$, $[\ ]$, and most especially subscripts; some facility in naming variables and in using symbols to designate constants not yet known explicitly.

3. Cultivation of expression of mathematical or near-mathematical ideas in reasonable sentences, aloud and in writing.

4. Rejection of some facts of life from models as being of lesser significance.

Apart from 2, these might be summarized as clarification, inclusion, conversation, exclusion.

B. Insight

1. Recognition that exact answers are not always obtainable and that experimental calculations may be useful.

2. Recognition of limits of applicability of models (see A-4), even when exact answers are obtainable.

3. Appreciation that not all useful insights are numerical.

4. Possibility of the same model form representing different situations.

Whether there are appropriate opportunities for the above, and whether they should be confined to separate sections of any new text materials or interspersed throughout, or both, can only be deduced after suggestions for class modeling activities are nominated and studied. Therefore, questions about how the teacher might proceed are not yet relevant.

We agreed that the essential thing is to give teachers examples of use of models as an encouragement for them to make more examples of their own and as an indication of our suggested policy for a writing group. Our first rough set of examples, as many and varied as we had time to produce, appears below.

We discussed the matter of censoring our examples to rule out those which are too specialized or too difficult or lack appeal to children. We emerged with the feeling that such censorship would be cowardly, unkind, and unwise. At most, there is a suggestion that where an expert in some field (e.g., operational analysis, physics) makes suggestions, he should be urged to censor his own suggestions, thinking whether they can be tailored to fit children in grades 7, 8, and 9. He should produce the simplest possible version as a starter for teachers, then the medium one which is nearer to his work, and, if he likes, a still harder one. At this point more suggestions seem better than a strictly censored list.

Examples of Topics for Mathematical Models

1. "Inside information" on tests (item analysis, norms, ranking, etc.).

2. "Inspirational" articles on mathematical models.

   Burington, R. S., "On the Nature of Applied Mathematics," A.M.M., (April, 1949), 56:221-242.

   Dantzig, G. B., "New Mathematical Methods in the Life Sciences," A.M.M., (January, 1964), 71:4-15.

   Gale, D., and Shapley, L. S., "College Admissions and the Stability of Marriage," A.M.M., (January, 1962), 9-15.

   Greenspan, H. T., "Applied Mathematics as a Science," A.M.M., (November, 1961), 68:862-880.

   Hamming, R. W., "Intellectual Implications of the Computer Revolution," A.M.M., (January, 1963), 70:4-11.

   Luce, R. E., "The Mathematics Used in Mathematical Psychology," A.M.M., (April, 1964), 71:364-378.

   Stone, M., "The Revolution in Mathematics," A.M.M., (October, 1961), 68:715-734.

   Wigner, E. P., "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," Communications on Pure and Applied Mathematics, (February, 1960), 13:1-14.

   Wilson, R. H., Jr., "The Importance of Mathematics in the Space Age," M.T., (May, 1964), 57:290-297.

3. Measure of all sorts; the "inaccuracy" of measuring instruments; discussion of plain and fancy techniques, and technology to reduce this inaccuracy; the role of improved technology and techniques in formulating and testing hypotheses.

   Beyers, M., "Are Earth-Measured Values Valid in Space?", Science and Mathematics Weekly, (February 20, 1963), 230-231; and (February 27, 1963), 244-245. (P. Peak, M.T., October, 1963, 56:448)

   Gager, W. A., "Computing with Approximate Data," School Science and Mathematics, (January, 1965), 68-83. (P. Peak, M.T., May, 1965, 58:438)

Rankel, P. J., "Quantification in the Social Sciences," M.T., (January, 1962), 55:20-33.

4. Various sorts of "ranking" under different assumptions. Eells, W. C., "One Hundred Eminent Mathematicians," M.T., (November, 1962), 55:582-588.

5. "Indirect measure." Fischer, I., "How Far is it from Here to There?" M.T., (February, 1965), 57:123-130.

6. Electoral vote. Polya, G., "The Minimum Fraction of the Popular Vote That Can Elect the President of the U. S.," M.T., (March, 1961), 54:131-133.

7. Distribution of prime pairs. Polya, G., "Heuristic Reasoning in the Theory of Numbers," A.M.M., (May, 1959), 66:365-384.

8. One-dimensional random walk. Foster, C., and Rapoport, A., "The Case of the Forgetful Burglar," A.M.M., (February, 1958), 65:71-76.

9. Landin, N. P., "Mechanics of Orbiting," M.T., (May, 1959), 52:361-364.

10. Tournament problems (bridge, tennis).

    Schied, F., "A Tournament Problem," A.M.M., (January, 1960), 67:39-41.

    Johnson, S. M., "A Tournament Problem," A.M.M., (May, 1959), 66:387-389.

11. Linkages. Stakes, G. D. C., "Linkages for the Trisection of an Angle and Duplication of the Cube," Proceedings of the Edinburgh Mathematical Society, (December, 1960), 1-4. (P. Peak, M.T., April, 1962, 55:250.)

12. Material on codes. Levin, J., "Variable Matrix Substitution in Algebraic Crypography," A.M.M., (March, 1958), 65:170-179. Also item No. 70 in this list.

13. Clifford, E. L., "An Application of the Law of Sines: How Far Must You Lead a Bird to Shoot it on the Wing." M.T., (May, 1961), 54:346-350.

14. deBethune, A. J., "Child Spacing: The Mathematical Probabilities," Science, (December 27, 1963), 1629-1634. (P. Peak, M.T., May, 1964, 57:354.)

15. Gale, D., and Shapley, L. S., "College Admissions and the Stability of Marriage," A.M.M., (January, 1962), 9-15.

16. Isaaks, R., "Optimal Horse Race Bets," A.M.M., (May, 1953), 60:310-315.

17. Old problems in astronomy, arithmetic.

Kennedy, E. S., and Hamaeanizadeh, K., "Applied Mathematics in Eleventh Century Iran: Abu Ja' Far's Determination of the Solar Parameters," M.T., (May, 1964), 58:441-446.

Kennedy, E. S., and Haydar, S., "Two Medieval Methods for Determining the Obliquity of the Ecliptic," M.T., (April, 1962), 55:286-290.

18. Klamkin, M. S., "A Moving Boundary Filtration Problem, or the 'Cigarette Problem,'" A.M.M., (December, 1957), 64:710-715.

19. Lambek, J., "The Mathematics of Sentence Structure," A.M.M., (March, 1958), 65:154-170.

20. Satellites.

Thompson, R. A., "Using High School Algebra and Geometry in Doppler Satellite Tracking," M.T., (April, 1965), 58:290-295.

Stretton, W. C., "The Velocity of Escape," M.T., (October, 1963), 56:400-402.

21. Combinatorial problems. Langlois, W. E., "The Number of Possible Auctions at Bridge," A.M.M., (August-September, 1962), 69:634-635.

22. Mudzi, J. S., "Probability and the Radio Active Disintegration Process," M.T., (December, 1961), 54:606-608.

23. Inheritance of blood type. Mulholland, H. T., and Smith, C. A. B., "An Inequality Arising in Genetical Theory," A.M.M., (October, 1959), 56:673-680.

24. Mathematics of the honeycomb.

Bleicher, M. N., and Toth, L. F., "Two Dimensional Honeycombs," A.M.M., (November, 1965), 72:969-973.

Seimens, E. F., "The Mathematics of the Honeycomb," M.T., April, 1965), 58:334-337.

Toth, L. F., "What the Bees Know and What They Do Not Know," Bulletin of the American Mathematical Society, (July, 1964), 468-481. (P. Peak, M.T., May, 1965, 58:440.)

25. Stoneham, R. G., "A Study of 60,000 Digits of the Transendental 'e'," A.M.M., (May, 1965), 72:483-500.

26. Stober, D. W., "Projectiles," M.T., (May, 1964), 57:317-322.

27. Hamilton lines and shortest network problems, by H. O. Pollak in a pamphlet published by Bell Telephone Laboratories.

28. Air defense theory.

29. Beginning of information theory and coding.

30. A napkin problem.

31. Radioactive decay.

32. Simple heat conduction by difference equations.

33. Simple genetic models. See number 23 above.

34. Optimum number of items in express check-out in supermarkets. (Discuss various criteria.)

35. Coverage of satellites by radar. See number 20 above.

36. Simple production scheduling over time.

37. Idea of PERT.

38. Assembly line balancing.

39. Elimination in baseball leagues.

40. Electoral college problem - knapsack problem.

41. Traveling salesman - machine scheduling. Ore, O., "Going Somewhere?" M.T., (March, 1960), 53:180-182.

42. Buying for several locations from several sources.

43. Queuing problems - simple arrival-service examples.

    Gramann, R. A., "A Queuing Simulation," M.T., (February, 1964), 57:66-72.

    Kendall, D. G., "Some Problems in the Theory of Queues," Journal of the Royal Statistical Society (1951), 13:151.

    *Morse, P. M., Queues, Inventories, and Maintenance, John Wiley, 1958.

    also Riordan's book.

*For problems primarily.

44. Markov chains - simple two- and three-state examples. *Howard, R. A., Dynamic Programming and Markov Processes, M.I.T. Press, 1960.

45. Coverage problems (two dimensional) - bombing examples, shoe-fitting problem.

Golomb, S. W., "Checkerboards and Polyominoes," A.M.M., (December, 1954), 61:675-682.

Robbins, A.M.M. (No specific reference)

Walkup, D. W., "Covering a Rectangle With T-Tetrominoes," A.M.M., (November, 1965), 72:986-988.

46. Present value of cash flows.

47. Smoothing and forecasting with linear functions. *Brown, R. G., Smoothing, Forecasting, and Prediction, Prentice-Hall, 1963.

48. Probability problems. Feller, W., An Introduction to Probability and its Applications, Wiley, 1950.

a) Sound measurement. Kryter, K. C., "Psychological Reaction to Aircraft Noise," Science, (March 18, 1966), 151:1346-1355.

b) Search theory. Koopman, B. O., Search and Screening, (OEG Report No. 56), N.D.R.C. Technical Report, Volume 2B of Div. 6.

c) Operations Research. *Morse, P. M., and Kimball, G. E., Methods of Operations Research (OEG Report No. 54); also printed by Wiley, M.I.T. Press, 1960.

Saaty, T. L., Mathematical Methods of Operations Research, McGraw-Hill, 1959.

49. Flow graphs.

a) Transition rates (or probabilities).

b) Representation of system of linear equations.

Notes on Operations Research, M.I.T. Press, 1959.

50. You have often subtracted numbers; for instance, making change at a cash register.

51. In chemistry the weights in weights are usually 1 gram, 2, 2, 5, 10, 20, 20, 50 grams. In grocery stores weights often run 1 ounce, 4, 8, 16 ounces, and so on. Which scheme is better?

*For problems primarily.

52. You do two problems, each leading to a lot of numbers

$$\frac{a \times b \times c \times d}{w \times x \times y \times z}.$$

In one problem you start with numbers like 18, 6, 4, 42. In the other, 19, 7, 5, 43. Your teacher tells you to cancel before you give her the answer. What can she tell more easily about your work from the second problem?

53. You know $(x^2 - y^2) = (x + y)(x - y)$. How can you use that to find 19×21 quickly?

54. Your class, 12 members in all, divides up into some equal teams to play a game, each team to play one match against each of the others.

a) There might be 6 teams of 2 each. Or their might be ...? How many different numbers of teams could you choose? What do you have to do (other than trying it out with coins) to answer that?

b) How many matches would be played for each of the team numbers?

c) Suppose there are only 11 in the class. Answer a) and b) again.

55. Six delivery boys deliver bread to a large block of apartments each day. Each boy delivers 7 loaves.

a) What do you do to the 6 and the 7 to find how many loaves are delivered in one day?

b) Do that.

c) Do the same with 6 boys per day and 7 loaves per boy. Does the result satisfy common sense?

d) Repeat c) but multiply if you divided in c), or divide if you multiplied in c). What can you make of the units of the answer?

56. (When e^{-x} is familiar) give radioactive decay and the intermediate model, with each child tossing a coin every $\frac{1}{4}$ minute. "Tails" retire from the game.

57. (Discussion) Provide a dozen pictures (e.g., of houses) and ask each student to rank the items from best to worst. Then discuss conflicting results by asking what criteria or assumptions were used.

58. My room has a doorway just 24 inches wide. My fat, round uncle has a circumference (by tape measure) of 72 inches. Can he get into my room? (Alternative: Pose the problem, limiting the measuring of uncle to a tape measure, and ask for method of solution.)

59. Family Christmas presents: Each of the N members gives a 10-cent present to every member including himself. Discuss total cost vs. N.

60. Graph degrees Centigrade against degrees Fahrenheit. Show Fahrenheit thermometer. Give a) a few pairs of integral readings; b) some with decimals. Discuss drawing a line. Draw it and use it. Discuss the limitations of real thermometer.

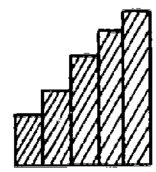
61. (As in PSSC.) Let "electric bell" device make marks on a paper strip drawn under it by

a) a student who starts, runs, stops;

b) something moving at a constant rate;

c) a falling object.

chart of strips pasted side by side



Calling the time interval a "tick," cut tape into 10=tick lengths and use them to make a chart (leading to a gradph).

62. Read a camera light meter at various distances from a small lamp. Hunt for $1/R^2$ law. Discuss effects of a) unwanted background illumination; b) using a long fluorescent tubular lamp.

63. Demonstrate switches that turn on this lamp only, this lamp or that, this lamp and that.

64. Show a millimeter scale on a wooden ruler and a good photograph of it, life-size. Then show the photo enlarged 2 times, 5, 10, and 50 times. Discuss the practical location of a point 1/3 mm from the zero mark, also points 0.3, 0.33, 0.333, ...

65. Look at a watch with a second hand.

a) If you had to use it to time a 100-yard race (about 15 seconds), how accurately could you time the race?

b) How much does a magnifying glass help? (Try one; argue.)

c) How would a tape recorder with two or more speeds of tape help? (magnifying time)

d) Listen to the ticking of your watch held close to your ear. How does that alter your claim?

66. Note: The physics of the following problem (S. H. M.) is too heavy to be suitable. This is given only as a suggestion for a type of extrapolation problem.

Predictions for a spring

The data below show the TOTAL LENGTH of a certain steel spring when carrying various loads.

LOAD (in pounds)	TOTAL LENGTH (in inches)
0	20
2	23
8	32

a) Predict total length with load 4 pounds.

b) Predict total length with load 12 pounds.

c) On what general physical knowledge did you base your predictions a) and b) ?

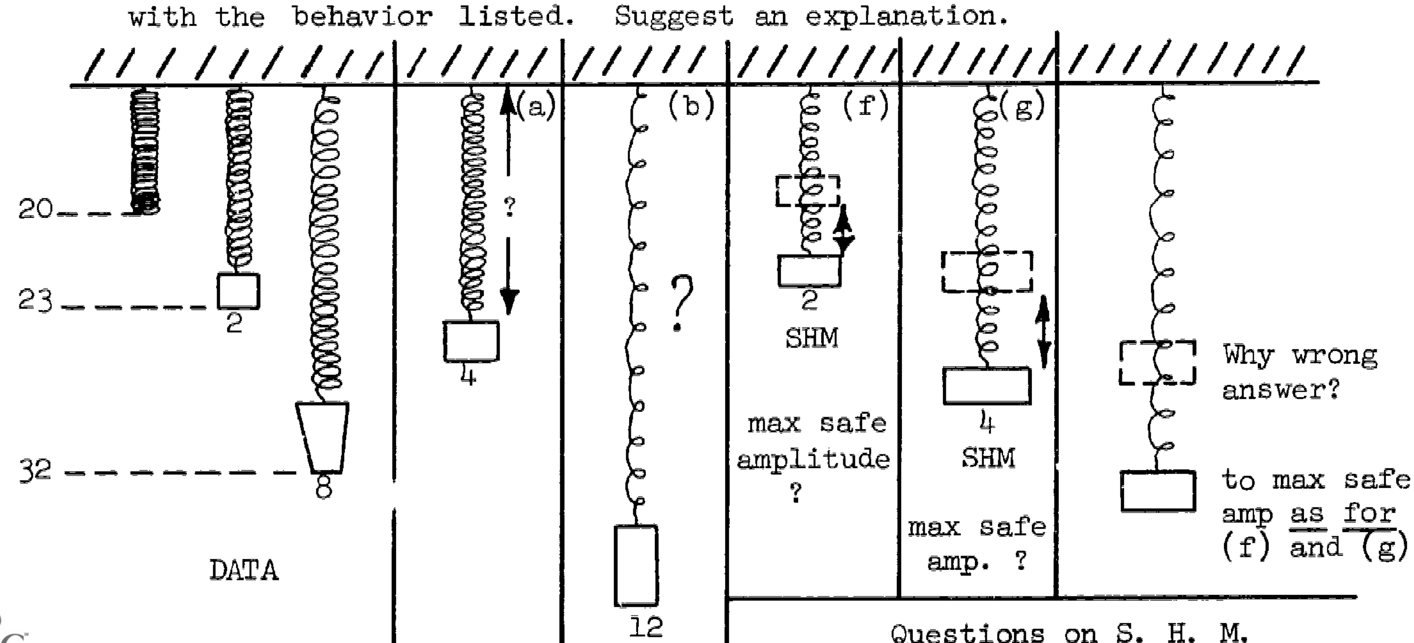
d) Do you consider your two predictions a) and b) equally reliable? Explain.

e) A 2-pound load is hung on the spring, pulled down below its equilibrium position and released to move up and down vertically. Why would you expect its motion thereafter to be S. H. M.?

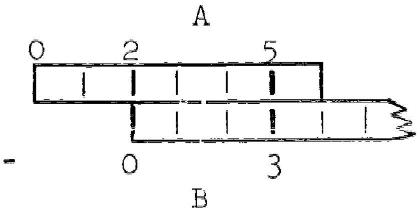
f) Predict the maximum amplitude (distance of load from rest position at each end of its cycle) for which the motion in e) would be perfect S. H. M. Give a clear reason for your prediction.

g) Predict the maximum amplitude for perfect S. H. M. with a total load of 4 pounds on the spring.

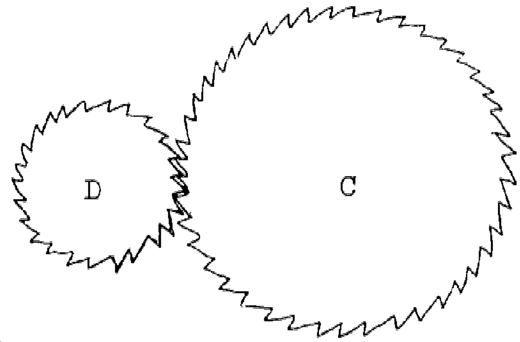
h) A theoretical physicist notices that the same line of thought is used in answering g) as in answering f), so he uses it to predict the maximum amplitude for perfect S. H. M. with a total load of 6 pounds. Experiment shows his prediction is wrong for a particular spring with the behavior listed. Suggest an explanation.



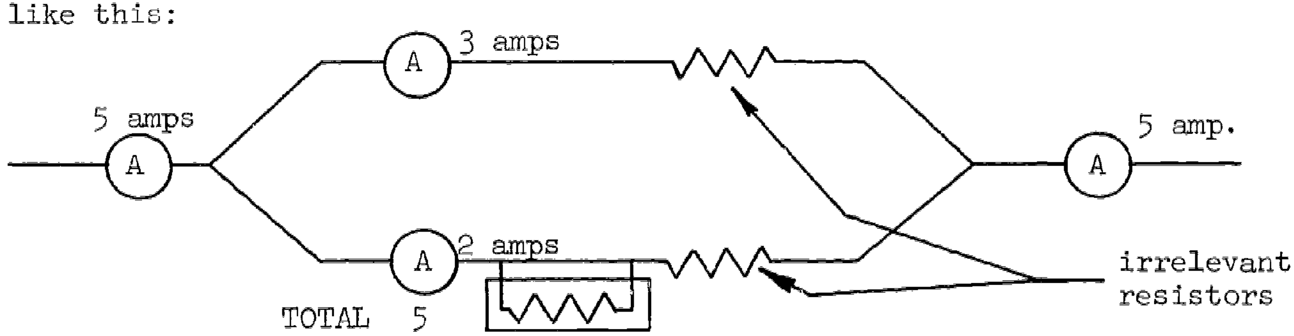
67. A and B are two rulers marked in inches. One slides along beside the other. At the moment, the beginning (zero) of B is opposite 2 on A, and the 3 of B is opposite 5 on A. What mathematical process is being carried out by this "calculating machine"? Explain how the machine is used.



68. C and D are two gear wheels of free axles. C has 4 times the diameter of D and 4 times as many teeth as D. If C is turned around 12 revolutions, how many revolutions does D make? Suppose there is a counter to count revolutions of C and another counter for D. What process can the machine carry out?

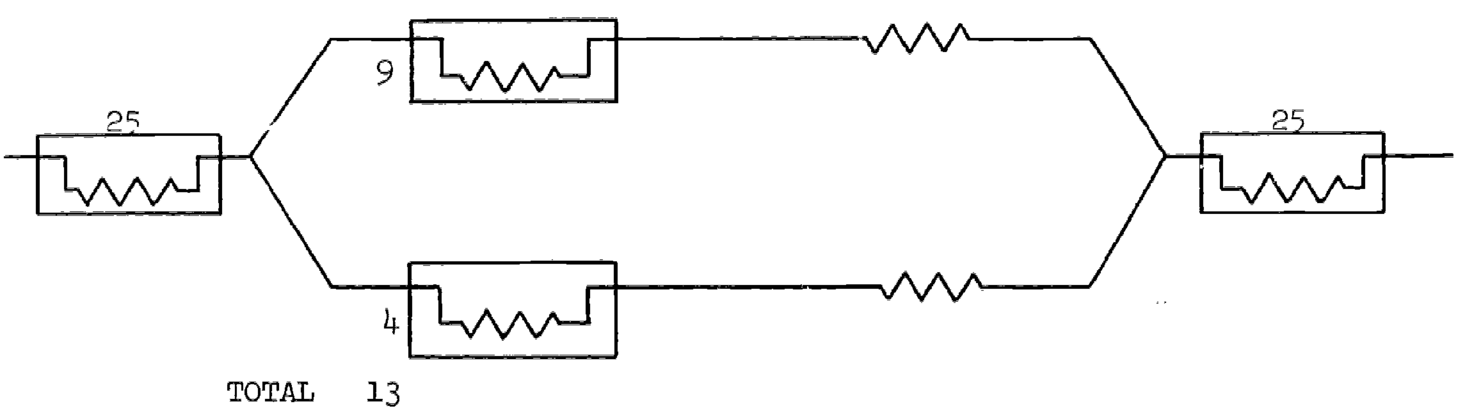


69. Suppose that all ammeters (A) in the (part of a) circuit sketched are exactly alike and in good order. Assume that electric currents keep to a constant total when they divide into two or more branches and that the total stays the same all around a circuit. (Otherwise they wouldn't have been called "currents.") The currents add up like this:



But if the resistances in boxes are all equal coils of the same thin wire, each coil in a pint of cold water, the temperature rises of the pints, when the currents run for 5 minutes, are as shown below. Trust that as true, and discuss.

Suppose ammeters had been constructed to <u>measure</u> electric currents in amps by using such temperature rises. Would that work well? Suggest.



70. Modified Caesar cipher

In the simple Caesar cipher, the encipherment is done by a fixed shift in the alphabet. For example, with key = shift left 4, one obtains the following:

plain text	USE ZORNS LEMMA
cipher text	QOA VKNJO HAIIW

This can be thrown into mathematical form (model ?) by corresponding numbers mod 26 to the letters, thus:

A B C D E F H I J K L M N O P Q R S T U V W X Y Z

0 1 2 3 4 5 . . . . . . 24 25

Operating in the arithmetic mod 26 , we write the equation for this cipher as

$$\text{plain} + \text{key} = \text{cipher}$$

$$p + k \quad c \ .$$

Note that solution of this equation yields the formula for decipherment: $p = c - k$.

Now, explore the possibilities of other cipher systems built on this same model and using elementary algebra. For example one might consider the system determined by the equation

$$c = 3p + 5 \ .$$

(Note practice in both elementary arithmetic and in working mod 26 which can be given by this example: Plain Q goes to cipher B because $p = 16$ yields $c = (3)(16) + 5 = 48 + 5 = 53 = 1$.)

Problem: How do you decrypt in this system? Idea: Solve equation for p, getting $3p = c - 5$, $p = \frac{1}{3}(c - 5)$. But what does $\frac{1}{3}$ mean? (Could also do simply: We return to the equation and solve by multiplying original equation by number chosen to simplify it, e.g., 9.)

Answer: $\frac{1}{3}$ is the solution, if there is one, of $(3)(?) = 1$. But, since $27 - 1$ and $(3)(9) = 27$, we have $\frac{1}{3} = 9$. Hence

$$p = (9)(c - 5) = 9c - 45 = 9c + 7 \ .$$

Problem: Can we use the cipher system described by

$$c = 6p + 5 \quad ?$$

Perhaps this looks okay. How do we decipher ? · $p = \frac{1}{6}(c - 5)$? Can we find a meaning for $\frac{1}{6}$? NO! What does this mean?

Answer: The system cannot be decrypted without ambiguity; both C and P (2 and 15) as plain text go into R as cipher. ONE and BAR both go into cipher as LFD.

Relevance: functions, correspondences, arithmetic rules, etc.

71. Digraph algebraic cipher

4	A	B	C	D	E
3	F	G	H	I_J	K
2	L	M	N	O	P
1	Q	R	S	T	U
0	V	W	X	Y	Z
	0	1	2	3	4

Each plain text letter corresponds to an ordered pair of numbers, as

$$D = (3,4)$$

We operate with these pairs, mod 5. If $p = (x,y)$ and $c = (u,v)$, then we have a cipher defined by the pair of equations

$$\left.\begin{aligned} u &= x + 2y \\ v &= 3x + 4y \end{aligned}\right\} \text{ mod } 5$$

For example, plain text $D = (3,4)$ goes into $u = 3 + 8 = 11 = 1$, $v = 9 + 16 = 25 = 0$, or $(1,0) = w$.

Problem: How do we decrypt this?
Answer: Solve for x and y.

72. Game problems. [Williams, J. D., The Compleat Strategyst, McGraw-Hill, 1954.]

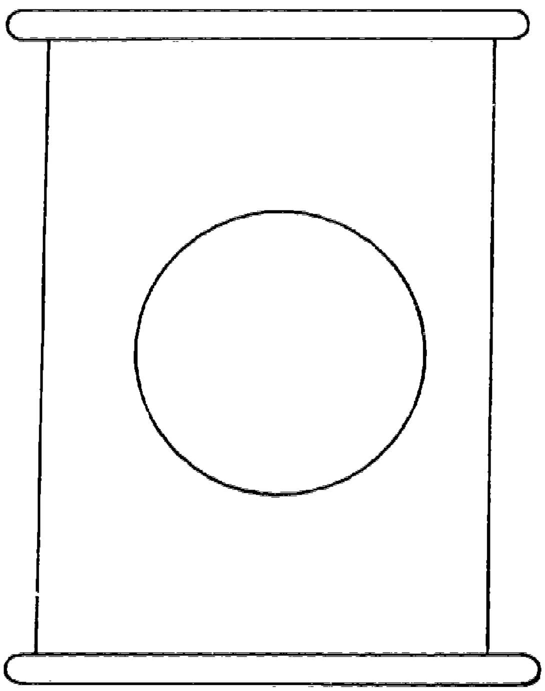
73. Ellipse application

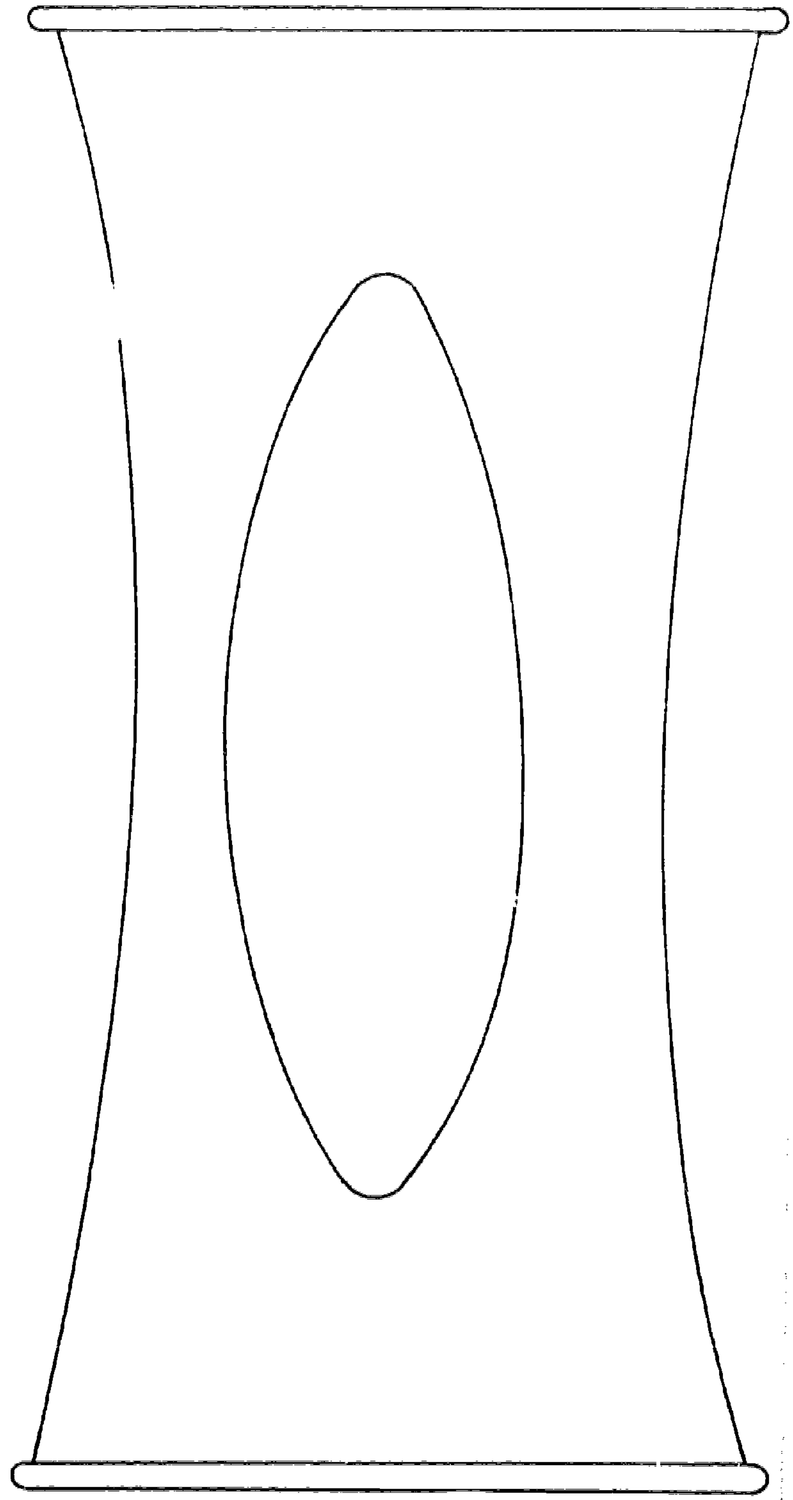
When a rubber sheet is pulled by forces applied evenly to its ends, the sheet stretches, with the length between those ends increasing, and contracts, with the width decreasing. When the length increases by 40%, the width decreases by about 10%. A circle can be described by $x^2/R^2 + y^2/R^2 = 1$.

Problem: A circle is drawn on a rubber sheet (unstretched) with a magic ink marker. The radius of the circle is 4 inches. The sheet, 10 inches long, is stretched to 14 inches.

a) Describe the shape that the circle now takes.

b) Write an equation for that changed shape.





Report of the Committee on the Real Number System

We suggested that the following chapters be included in the materials for grades 7-9:

1. Number theory
2. Negative numbers
3. Rational numbers
4. Decimals
5. Radicals, etc.
6. General properties of real numbers

We suggested that a chpater on functions and graphing come after Chapter 2, a chapter on probability after Chapter 3, and a chapter on algebra after Chapter 5.

Remarks and Suggestions

It seemed to us that a good topic for the start of grade 7 would be elementary number theory. Prerequisite: arithmetic of the non-negative integers.

The objectives are to acquire some familiarity with the multiplicative structure of the positive integers, to drill on elementary arithmetic, to learn something about (informal) proof, and to acquire some techniques to be useful later.

Sample Topics

primes and factorization (note need to introduce exponents)

sieve process, proof of nonending of sequence of primes

prime-generating algorithms: factor 1 + product of primes; computable; prime factors of $2 + 1$, $2^2 + 1$, $2^4 + 1$, $2^8 + 1$, $2^{16} + 1 \ldots$, which are all distinct; proof (perhaps too hard)

divisibility tests, other special divisibility problems

gcd and Euclid algorithm (again computable)

This area is rich in examples; see papers by Rosenbloom, Fine. Explore phenomena: residues of powers, of the Fibbonacci sequence, modular arithmetic, etc.

To handle some of the arguments, one will have to take care in the explicit use of "and," "or," and "not" on an informal level.

We suggest that the arithmetic of negative numbers (integers and rationals) be introduced by means of something like the vector approach proposed by Moredock and Sandman. This gives an excellent example of modelling, good motivation, and lots of practice in working with calculations. It is also good preparation for the planar vectors that will be done a little later on. Prerequisite: Negative integers have been assigned places on the number line.

The subject to be studied is trips on a line (not on the number line). A unit of distance is chosen. Geometrically, a trip is an arrow (vector) whose length represents the length of the trip, and direction is either left or right (i.e., which way the arrow points). We add these geometrically. Now we want a mathematical model for this. As a first step, we use left 6 and right 6. We use 0 for "no trip" or, better, "zero trip

Drill on the operation of adding trips.

left 5 + left 7 = left 12

left 5 + right 9 = right 4

left 5 + right 3 = left 2

right 5 + left 2 = left 2 + right 5

right 6 + left 6 = 0

0 + right 18 = right 18

Introduce the operation (function?) "opposite."

opp(right 5) = left 5

opp(left 4) = right 4

Observe: opp(opp(anything)) = that thing.

Formalize as a law (theorem?): opp(opp V) = V .

Do more on the arithmetic of vectors, i.e., associative law of addition, commutative law of addition:

opp(right 6 + left 3) = opp(right 6) + opp(left 3)

= left 3

Solve problems:

right 7 + V = left 2

$$\text{left } 7 + \text{right } 7 + V = \text{left } 7 + \text{left } 2 = \text{left } 9$$

$$0 + V = \text{left } 9$$

Replace vector model by use of positive and negative numbers:

$$\text{left } 5 \Longrightarrow -5 , \quad \text{right } 6 \Longrightarrow 6 , \quad 0 \Longrightarrow 0$$

Drill on arithmetic problems with this context: Pupils may return to vector form if they wish. Solve equations: $-6 + A = -2$ by adding 6 to both sides. Solve $7 + A = -2$ by adding -7 to both sides. (Do this with rational numbers too? decimals too? Why not?) Return to vectors and introduce scalars as multipliers. "Double the vector --------> is ------------------> ", etc. "One third of the vector <-------- is <--". Go over to the first formulation. Twice (left 4) is left 8. $2 \times$ (left 4) = left 8. $3 \times$ (right 11.4) = right 34.2 . Observe distributive law in simple cases. (Talk about scale change here? New units?) $0 \times V = 0$.

Reformulate in terms of the numbers:

$$2 \times (-4) = 18 , \quad 3 \times (1.3) = -3.9 .$$

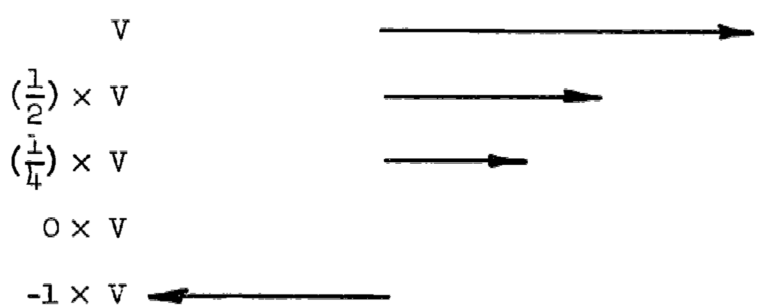
Observe: $3 \times (-1) = -3$

Want: $-1 \times 3 = -3$ (universal commutative law)

That is: $-1 \times$ (right 3) = left 3

Discover: $\text{opp}(V) = (-1)V$ (for some V) Drill?

Geometric argument to make this look reasonable:



Meaning of: $(-2) \times 3$? Answer: same as $3 \times (-2) = -6$.

Observe: $-2 = (-1) \times 2$. $(-2) \times 3 = (-1) \times 2 \times 3 = (-1) \times 6 = 6$. (Universal associative law)

$(-1) \times (-5) = ?$ $(-1) \times$ left 5 = opp(left 5) = right 5

Hence, $(-1) \times (-5) = 5$. Again, $(-1) \times V = \text{opp}(V)$. Lots more drill on this.

Decimals, terminating and otherwise

Students should either do, or see done on a desk calculator, some divisions carried out far enough to see the periodicity of simpler rational numbers. Do we use $.333\frac{1}{3} = .333333\frac{1}{3} = ?$ Ideas of nonperiodic, nonterminating decimals are apt to be very fuzzy at this level (as they are with Brouwer). Can't we leave them that way? Square roots can be studied and calculated in approximate form. Here is a chance to do more algorithms, more approximation. The irrationality of $\sqrt{2}$ can be done since there is enough number theory available. Problem: How to add too infinite decimals? Worse, how to multiply them? Can these problems be raised?

Real numbers

Obviously, there must be some time spent on statements (or sentences) and some experience with these before one is to present a formalized postulate system for an ordered field (or for a field), e.g., the real numbers. Recommendation: Exploit this linguistic skill in formulating careful, precise statements about the real numbers (i.e., aim at the description of a field), and about the integers (i.e., a ring). Make no claim about categoricity, but indicate that there will be other systems that obey these laws too. As an exercise in proving things, show now that the laws for fractions and the fact that $(-a)(-b) = ab$ follow from these formulated principles. (Is this at ninth grade level?) Also, observe that you can prove $ab = 0$ implies $a = 0$ or $b = 0$ from the field postulates, but don't seem to be able to form the ring postulates. (Use mod 12 and mod 7 to illustrate?) (Honor work: $a + b\sqrt{2}$, where a,b rational, is a field.)

Idea: The order might be: linguistics and mathematical sentences, etc., formulating of field and ring axioms, all of linear equations, word problems, etc., then theorems about fields and rings, then quadratic equations, then finite fields, etc., then complex numbers.

Note: In the discussion of this report, there was a shart disagreement on the treatment of multiplication of negative numbers.

In the SMSG First Course in Algebra, the treatment is based on the distributive law and the desire to have it continue to hold for negatives as well as for positives. In the elementary school mathematical concepts about whole numbers and rationals and operations on them are extracted from concrete situations. On the other hand, at some stage the development of a new number system (e.g., complex numbers) will be based on abstraction from a previously developed number system and not from the real world. In the SMSG program this transition takes place in the development of the arithmetic of negative numbers.

Some members of the conference felt that the mathematical treatment should be accompanied, rather than followed, by various physical or geometrical situations which suggest how negatives should multiply.

It was agreed that this important point should be subjected to careful classroom experimentation.

Report of the Committee on Geometry and Measurement

ume students are somewhat acquainted with certain elementary clas... of subsets of E^1, E^2, and E^3, line segments, polygons, circles discs, spheres, cones, etc.

Topics

1. Develop intuitive ideas of certain motions built on physical motions: translations, reflections, rotations. Any others?

2. New classes of subsets. Convex? If so, why? Others?

3. New figures from old: a) Decomposition and composition: cutting and slicing, tiling, cartesian products; b) projection and perspection.

4. Relations between figures: congruency, similarity, projectivity, perspectivity, relations under maps.

5. Relations and connections between algebra and geometry: coordinatizing of spaces, sets, and maps.

6. Measurement: what measured (lengths, areas, volume of certain sets, measure of angles), fundamental properties of measuring function, extension of measure function.

7. Axiomatics. Up in the air, except agreed that some significant chuck should be accomplished by the end of the ninth grade. What? When? How much?

General procedure

Physical objects and movements to drawings to abstractions.

Report of the Committee on Algebra

[Note: The original report of this committee suggested that the treatment of algebra in the SMSG text First Course in Algebra be retained (but with more applications to real problems), noting that Chapters 1-9 would have been covered in the recommendations of the Committee on the Real Number System. In the discussion of this report by the entire group, the treatment of open sentences in Chapter 4 was questioned, leading to the appointment of the committee whose report appears next.

Also, the group agreed that the algebra of polynomials and rational expressions should be postponed until after grade 9. No agreement was reached on how much of this was really needed in grades 10-12.]

We recommend that the function concept be used throughout.

We recommend that the following topics be included in algebra in grade 9, treated as in the SMSG text First Course in Algebra:

1. Factoring and exponents.

   a) A little number theory

   b) A little informal ideal theory to get gcd

2. Radicals (particularly square root): Approximation techniques with relation to computing, flow charts, etc.

3. Graphs of functions

   a) Linear

   b) Lead to change of coordinate systems by analysis of $y = mx + b$ under various b, m, etc.

4. Systems of equations and inequalities: linear programming

5. Change of coordinates (translations only)

6. Quadratic functions

   a) "Completing square"

   b) Finding minimum

   c) Changing coordinates

7. Statistics: least squares fit.

8. Vectors

a) By displacement arguments

b) Re-emphesis of algebraic properties showing similarity to R#.

Most parts of linear algebra would have to be delayed until rotations (hence trigonometry) are available. Then matrices used in change of co-ordinates would be natural.

Report of the Committee on Open Sentences

As the situation stands now, it turns out that there is no intrinsic property of a sentence by means of which one may determine whether or not it is open. If a sentence is open, then so is any sentence derived from it by those permissible operations on sentences leading to equivalent sentences.

For example, in the expression

$$\{x \mid x + 1 < x + 3\}$$

the sentence "$x + 1 < x + 3$" is an open sentence. And the equivalent sentence "$1 < 3$" is an open sentence in

$$\{x \mid 1 < 3\} .$$

Hence we cannot properly make a distinction between "open" sentences and other kinds, and "open sentence" must be synonymous with "sentence."

It is therefore suggested merely that the adjective "open" be dropped.

We could then consider the truth set of the sentence, say,

$$x + 9 < 2x + 7$$

(and it is suggested that the notation

$$\{x \mid x + 9 < 2x + 7\}$$

be used, as the students taking this course will be familiar with set notation.) We would agree that a particular number, say 5, is in the truth set if when we assign the value 5 to x (language borrowed from flow charting), the statement

$$x + 9 < 2x + 7$$

becomes a true statement. Then we must consider examples such as

$$\{x \mid 1 = 1\} \quad \text{and} \quad \{x \mid 3 > 7\} .$$

We do this not to pick nits, but because such situations will invariably arise when our initial statements are always true (identities) or always false.

The upshot of all this is that we do not need to classify sentences according to whether we can determine the truth or falsity without more information.

There should certainly be no objection to mentioning informally that in certain sentences, like $1 = 2$ or $1 \neq 2$, no variable appears, so that the truth or falsity does not depend on the value assigned to this variable. In other sentences, such as $x \neq 9$, a variable does appear, and the truth does depend on the value assigned to that variable. Finally, there is a sort of never-land of sentences, such as

$$"x = x" \quad \text{or} \quad "x \neq x" ,$$

in which a variable appears, but whose truth does not depend on the value assigned the variable. Even more finally, there is the never-never-land of sentences like

$$x = [x] ,$$

which is universal over the set of integers but not over the set of real numbers. We see no purpose in dragging in this last example.

We should follow the excellent procedure of FCA (SMSG 9) for translating from English into mathematics. But once this has been done, we recommend suppressing the phrase "truth set of the sentence" in favor of "solution of the equation" or "solution (or solution set) of the inequality." Still we should return occasionally, say once every week or two, to the "truth set of sentence" terminology in order to remind the student of the meaning of "solution."

The changes we recommend are very slight.

Our group has one recommendation concerning variables, namely, that the concept of variable in the algebra be "related" to the concept used in flow charting.

Report of the Committee on Functions

A) Functions will be introduced early in the seventh grade and used throughout.

B) Extrapolated game situations can be used profitably during the introduction of the concept of function.

C) The important thing to be learned is the association and not the way it is described. Avoid defining functions as sets of ordered pairs. Use formulas, graphs, tables, etc.

D) Domain should be unrestricted, but the range will normally be sets of numbers or other mathematical objects in order to avoid artificial situations.

E) Functions t be considered:

- a) Look at old binary operations ("add 4", etc.)
- b) Look at old unary operations: "oppositing," right shifts, etc.
- c) Linear functions
- d) Step functions
- e) Absolute value
- f) Greatest integer (to avoid leaving the impression that all functions are neatly describable)
- g) Quadratic functions

F) Since it had been suggested that the trigonometric functions be introduced early, the following "flow chart" may be of interest.

absolute value → distance in plane ← radicals

distance in plane

↓

combined geometric and analytic way of looking at circles

↓

↓

trig functions on circle

↓

complex numbers, rotations

↓

second round quadratic functions

Report of the Committee on Probability

Why?

a) Subject is important in its own right.

b) Provides an excellent medium for learning and reinforcing concepts and techniques in other areas of mathematics, e.g., rational numbers in decimal and in fractional form, union and intersection of sets.

c) Applications are interesting to students and cover a wide range of ideas from easy to difficult.

What?

Probability

Certainty vs. uncertainty, numerical measure of uncertainty. Equally likely outcomes, mutually exclusive, independent. Conditional probability. Binomial distribution.

Statistics

Mean, median of sample data; how data affect them. Some discussion of spread. Representation of data (technical discussion of inference to occur after ninth grade).

When? Start in seventh grade and continue through ninth.

How?

a) Students should do lots of experiments.

b) Much of the discussion should be intuitive and informal.

c) Students should argue through from first principles, not just memorize formulas.

d) Include significant examples of the use of probability; for illustration, problems beyond the scope of the students.

Report of the Committee on Algorithms, Flow Charts, Computing

It is proposed not to teach any computing or use of computers but to confine our activities along these lines to algorithms and flow charts. For the purpose of titillating him, we would tell the student that making a flow chart is the essence of preparing a problem for a computer. We recommend that care be taken to teach flow charting only as it is useful in clarifying the algorithms to be taken up at these grade levels; we should not get side-tracked into teaching flow charting for its own sake. We value as a by-product the knowledge of flow charts and algorithms the student will have acquired for future use in computing.

Aims and aspirations

If we adopt an optimistic attitude, there are numerous advantages which might accrue through introducing flow charting in grades 7-9. The geometric or diagrammatic lay-out of flow charts will aid the student in understanding algorithms, for then flow charting techniques will help the student devise his own algorithms for solving problems. The method might help overcome the block so many students have against starting to work on a problem when they don't see how to do it from beginning to end. Flow charting also provides application and reinforcement of the function concept.

Prerequisites

The arithmetic operations are of course prerequisite and also variables, at least in the sense of using letters to stand for numbers. Functions are needed in some but not all flow charts, in particular, the greatest integer function. Some examples are contemplated using number bases other than ten.

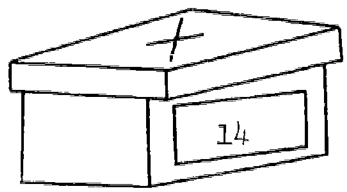
Implementation

We suggest that some "brain busters," such as cannibal-missionary, coin weighing, and concentration camp problems, be put at the ends of the early chapters of the seventh grade. Later, toward the end of the seventh grade, their solutions could be examined and given as axamples of algorithms. Now we follow _Algorithms, Computations, and Mathematics_, the material being taken up being a subset of Chapters 2 and 3 of the book. We define an algorithm as an unambiguous plan (usually involving iteration) for carrying out a process in a finite number of steps.

Variables and assignment

In flow charting, a "variable" is a letter (or other symbol) to which numerical values may be assigned. At any given moment each variable in the flow chart has a definite value. When we see a variable in a formula in the flow chart, we treat the variable as a name for its (present) value.

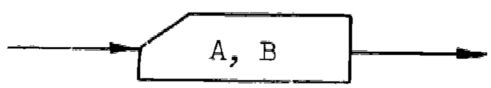
It simplifies matters conceptually to consider that there is associated with each variable used in the flow chart a box with a window and with the variable engraved on the cover. When we assign a value to a variable,



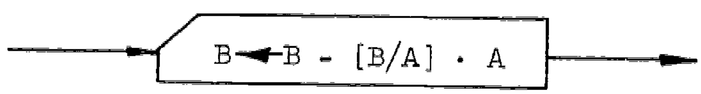
we open the box, dump out the old value, and put in the new one. When we need the value of a variable in a computation, we read the value through the window so as to be sure not to alter the value.

Components of algorithms and flow charts

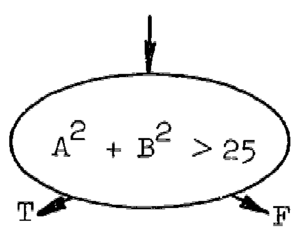
The components of the algorithms treated would be sequential steps, branching, and iteration. The components of the flow charts would be the input box,



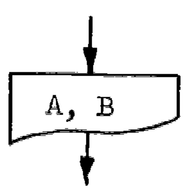
the assignment box,



the decision box,



and the output box.



(There might be an exception of one or two introductory flow charts of human activities, such as a flow chart for getting up and getting dressed in the morning or for robbing a bank.)

The instructions in the above assignment box mean that the value of $B - [B/A] \cdot A$ is to be computed and this value is to be assigned to the variable B. ([ ] denotes greater integer function.)

Group activities

Early in the game the class should "trace out" some flow charts with the window boxes actually provided and some students given in effect the role of part of the computer. (See ACM, end of Section 2-4.)

Individual activities

The student will be asked to trace out some flow charts thus marking the indicated computations. Later he will be given algorithms and asked to construct flow charts for them.

Still later he will be given problems and asked to devise both algorithms and flow charts. (Here, incidentally, he will learn that even though a problem may have a unique solution, it will not have a unique method of solution.)

The student should be asked to take some flow charts home and, without explaining what algorithms they represent, to have some member of his family trace them out. Thus the student will learn that the instructions in a flow chart can be executed even though the executioner doesn't know what the flow chart represents. This suggests the possibility of machine execution.

It may or may not be advisable to analyze some flow charts; i.e., he is given the flow chart and asked to discover the algorithm.

Samples of algorithms to be treated

1. Euclidean algorithm

2. Square root "divide and average" algorithm. Also include traditional square root algorithm as a horrible example. Discuss the difference.

3. Cube root by divide and average.

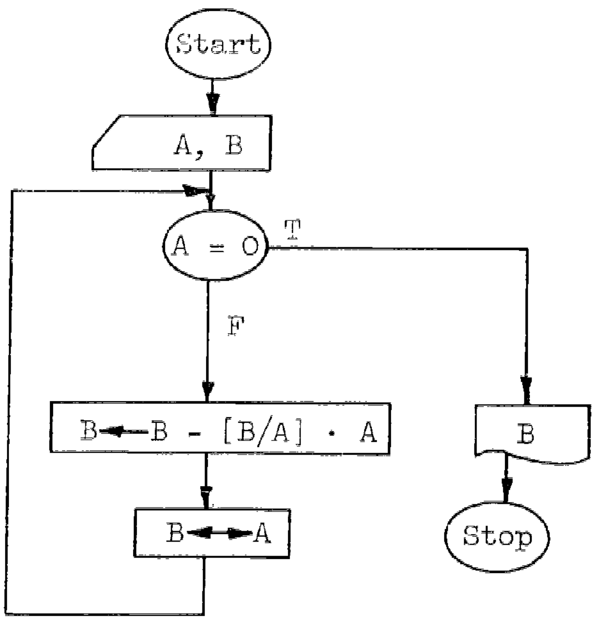
4. Ordinary algorithms for "long" addition, subtraction, multiplication, and division showing borrowing and carrying.

5. Least common multiple

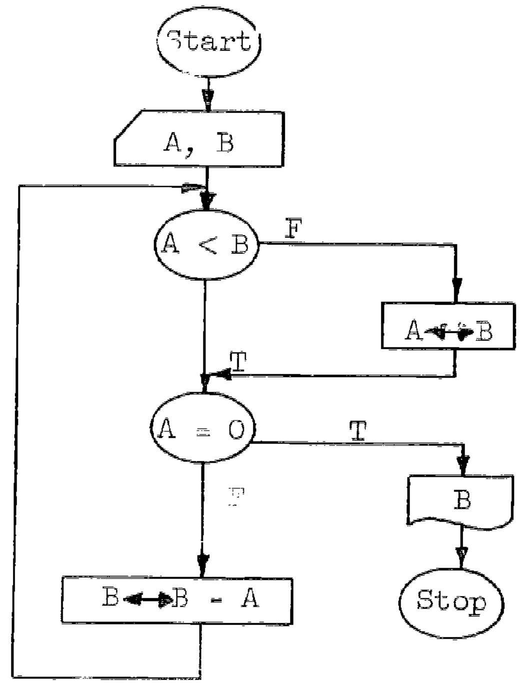
6. Complete factorization

Sample Flow Charts

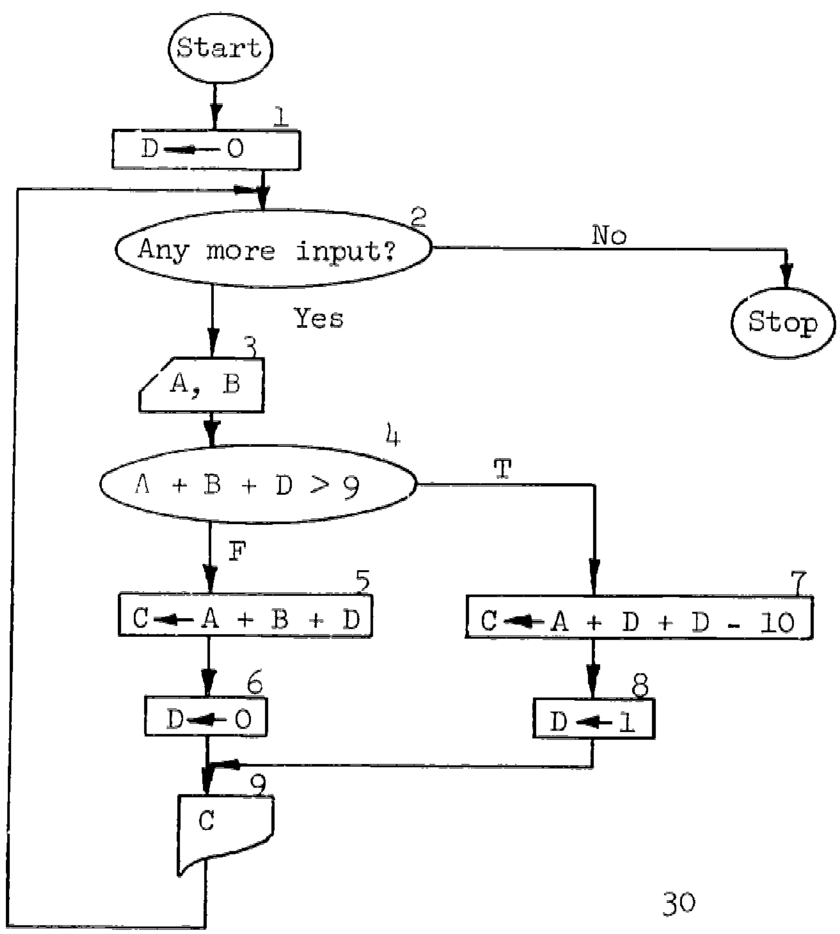
Euclidean Algorithm (1)



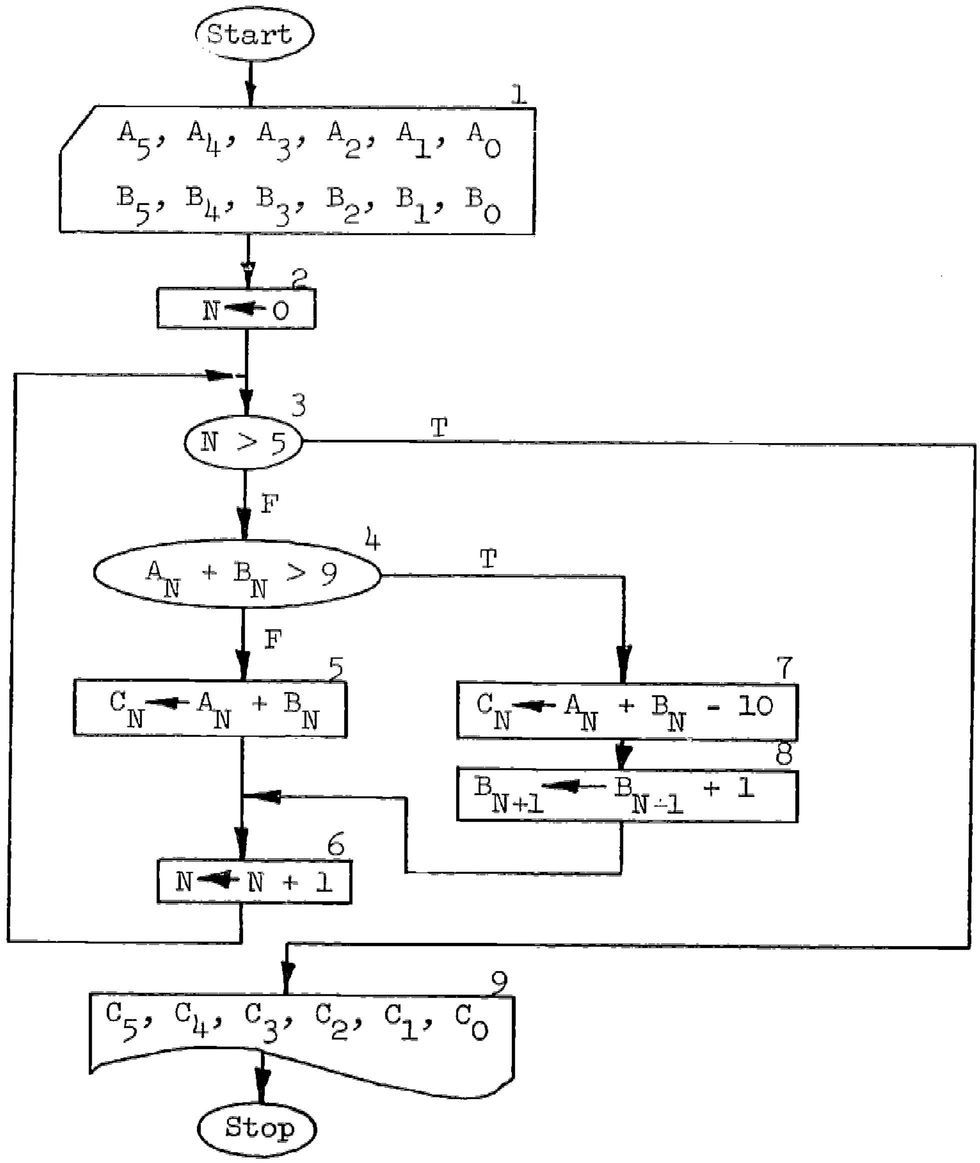
Euclidean Algorithm (2)



Long Addition Algorithm (1)



Long Addition Algorithm (2)



Report of the Committee on Reasoning

Since mathematics is, essentially, the carrying out of logical reasoning, a primary concern in the teaching of mathematics must be the manner in which the student learns such reasoning. Yet if there is too much emphasis on the logic itself, the understanding and easy thought about the mathematical objects themselves will be hindered. Best is for the student to be drawn into logical reasoning through its helpfulness in solving intriguing problems, then to start considering the type of reasoning he has used. This should certainly be a serious concern in the seventh grade. An ideal subject for this training is number theory. We shall point out some of the ways in which this training may be carried out.

By "number" we shall mean either positive integer or integer. In writing up an actual text, one must decide to what extent negative numbers will be used; perhaps they can be introduced in an intuitive manner for the present purposes.

The seventh grade student has had much training in arithmetic. But his understanding of divisibility is meager. The question "Is 51 the answer to a multiplication problem?" may be new to him. If the answer is yes, then 51 is composite; otherwise, it is prime. This now suggests two theorems: 7 is prime; 9 is not prime. For the first proof, try 2, 3, 4, 5, 6; for the second, see that $3 \cdot 3 = 9$. One can now restate the definition of prime number and see that an actual proof has been carried out. Thus the use of quantifiers, and formation of negation, is already made somewhat explicit, in a manner that the student feels he already understands. This illustrates a basic point: Examining any tool, logic in particular, should as far as possible be simply a reinforcement of what the student feels he realizes already.

To consider divisibility, try some examples. What numbers are divisible by 2? By 4? Let us look at the pattern:

$$1 \quad \cancel{2} \quad 3 \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad \cancel{8} \quad 9 \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \ldots$$

Obviously if a number is divisible by 4, it is divisibly by 2; but not conversely. Perhaps a week later, this concept may be studied: There are two simple interpretations. As sets of numbers, one set is a proper subset of another. In a logical expression,

$$(\forall x)[\mathrm{div}_4(x) \rightarrow \mathrm{div}_2(x)],$$

$$\text{not-}(\forall x)[\mathrm{div}_2(x) \rightarrow \mathrm{div}_4(x)].$$

The second is equivalent to:

$$(\exists x)[\mathrm{div}_2(x) \text{ and not-}\mathrm{div}_4(x)].$$

So far, any use of shorthand notation (even the use of symbols to name numbers) is mostly for ease in seeing pictured in a short form what you know. At present, it must not interrupt the discovery of properties of numbers.

Is 102 divisibly by 6? Yes; you can divide by 2 and then by 3. This suggests: n is divisible by ab if and only if it is divisible by a and by b. Trying this out soon shows difficulties. For what a and b does it hold? Finally, it is apparent that they must have no common factor. A considerable use of logic is inherent even in this statement; the student may view this (with the teacher's help) in both the set theoretic and logical form, though not yet with logical notation.

How do we find out if 53 is prime? By definition, we test 2, 3, ..., 52. Clearly we wish to shorten this. Why do we not need to try 49? Or 29? The class may search for the reason for this. Analyzing it carefully, we see that 7 is the last number we need try. (Note that this is a real mathematical investigation, much in the manner of research.)

Why do we need not try 4 or 6? More logic is involved here. This could be a good time to introduce the sieve; one can actually count the primes less than 50, considering 49 as a special case.

To return to a simpler situation, let us show that if n is even, then n^2 is even. For if $n = 2k$, then $n^2 = 2(2k^2)$. We know that n is odd if and only if it is not even. Now show that if n^2 is odd, then n is odd; proof by contradiction. Carrying this through carefully enough (go through the proof in the opposite direction) shows a basic logical method (without need of fancy terms like "contrapositive"). One can also find $(2k)^2$ and $(2k + 1)^2$, thus finding a fuller picture, and relate this to the former proofs.

Now one can show that $\sqrt{2}$ is irrational. Note first the translation into better terms: There are no numbers a, b such that $(a/b)^2 = 2$. (Time to discuss the meaning of "number"!)

This proof shows a very basic point: During the proof we carry out certain operations of arithmetic on numbers; yet at the end we see that there were no such numbers. What can the meaning be? We worked with the numbers <u>under the premise</u> that we were given them. Thus a statement has meaning only in terms of previously given things.

Quotients and residues is a nice topic to take up here. If we look at multiples of a mod 10 for instance, we put, on a circle, dots for a, 2a, 3a, ... Trying this for various n and a, we see interesting patterns emerge. Every pattern seems to end up evenly distributed. Is this always the case? Why? The brighter students will right away want to see a complete proof; thus "proof" has its best meaning here: complete argument convincing the reader. Other students ·ll be sufficiently convinced by the pictures and thus be able to use t result. We have found the gcd of a and n.

At times, here and there, one will have noted how "and," "or," "if... then...," etc., have been used; they are becoming part of the students' working tools. Statements and truth values, in particular, are taking on meaning.

The basic property of primes--if p divides ab, then it divides a or b--may now appear. From it we soon have the fundamental theorem of arithmetic. It is now fun to use exponents and write numbers in symbolid factored form:

$$(2, 0, 1, 2) \text{ means } 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^2 = 4 \cdot 5 \cdot 49 = 980, \text{ etc.}$$

Multiplication and division are easy; divisibility translates in a pretty way. How about squares and square roots? Obviously we find general theorems on roots.

In later grades one can take up some of these proofs again and consider more carefully the logic involved. By the tenth grade the student should be able to analyze a good proof quite completely, checking it just as a mathematician might. Just how the purely logical study is introduced must be left to the actual writer of the given texts.

A very basic question to be considered is how symbols are introduced. For the best exposition, a symbol never appears without explanation; normally, one wishes to prove $(\forall x)$..., so one takes any x from the domain; or one has $(\exists x)$... and so has an x given. The usual habit of "looking at an equation" (or open sentence, etc.) to see its meaning should be unnecessary, since it should already have been explained.

The insufficiency of chapters on logic in ordinary texts is shown by one example:

$$\text{If } x \neq 0, \text{ then } \frac{x}{x} = 1.$$

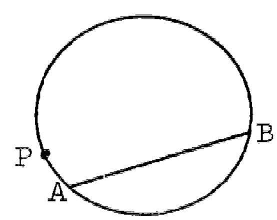
For instance, one should deduce (?) if $\frac{x}{x} \neq 1$, then $x = 0$. Of course, here both use of symbols and meaning of "if...then..." comes into question.

Random Additional Comments

As suggested by Rosenbloom, Dilworth, etc., there should be explicit consideration given to Polya's strategies for doing problems.

The text should include some problems which have easy solutions (once you have seen them) and which will lead students to ask, "Why did you think of doing that?" To which the teacher should consider giving the answer, "I have no idea--nor do others--as to why the first person to do this did that."

Example:



Construct a chord through P that will be bisected by AB.

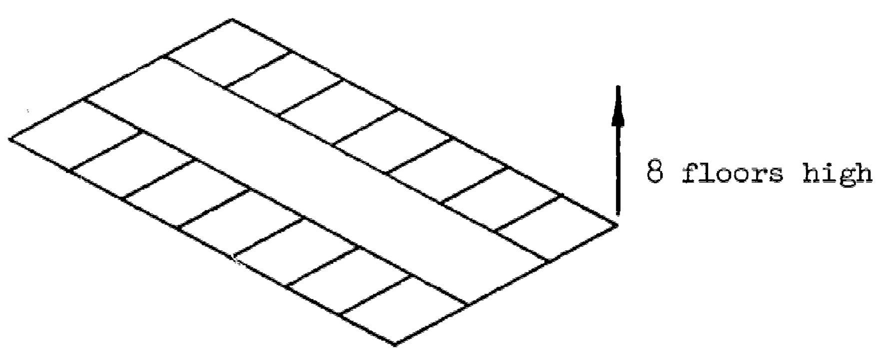
Can we teach the students and teachers to use the word "conjecture"?

There might be in the text questions for the student to ask his teacher. There might also be problems to take home for family discussion.

Good questions to develop reasoning? Open-ended, some easy parts, some hard, e.g., When is a pin jointed structure rigid?

How about group problems (team solution)? Which is better for auto plates: 5 numbers, 5 letters, or a mixture?

(Stolen from the British Book T, page 253) What are some ways to number the apartments in this building?



Invent a simple game and see if they can analyze it. (Take turns coloring an edge in a specific polygonal comples; the winner is the player who first gets a triangle colored in his color.)

Number theory problems that lead to discovery of regularities, e.g., remainders of the Fibonacci, or numbers that are the sums of two, three, or four squares.

Or, how many triangles can you make from sticks of length 1, 2, 3, ... 8? How many tetrahedra? Etc.

Or Wisner's game: Take the sum of the squares of digits of numbers and repeat.

Miscellaneous Suggestions

In the course of various discussions, a few suggestions for later consideration were put forth.

Infinite Decimals

In the development of the real number system, infinite decimals will surely be discussed. Ordinarily, however, the problem of performing the arithmetic operations on infinite decimals is slighted. It is suggested that such a discussion might be motivated, after students have learned how to approximate square roots, by problems such as:

> Compute $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ to three decimal places. Multiply the first two and compare with the third.

Slide Rules

It was suggested that the early use of slide rules (and desk calculators ?) be considered, even though the basic principle cannot be explained, as a way to develop ability to approximate and as a way to force consideration of order of magnitude.

Vectors and Matrices

It was suggested that these concepts might be introduced informally at an early stage as methods for organizing data and displaying relationships.

Mathematics in Grades 10-12

Less time was devoted to a discussion of a mathematics program for these grades than had been devoted to the program for grades 7-9. It was agreed, with very little discussion, that the work on probability started in the earlier grades should be continued in these grades. It was agreed not to attempt to settle the question as to whether any work on statistical inference should be included.

It was also agreed, with very little discussion, that further work on algorithms and flow charting should be inserted at appropriate places.

Committees were asked to recommend content for grades 10 through 12 from the areas of geometry, linear algebra, and calculus. In addition, a committee was asked to consider the architecture of the over-all program and the spirit in which it should be presented.

Report of the Committee on Geometry

Prerequisite: The sort of non-formal, rich geometric experience mentioned in the earlier report.

Objective: Provide some experience in systematic axiomatic deduction; more experience in constructing and understanding proofs; give the students a sampling of the more interesting (and useful ?) theorems about triangles, circles, etc. ["sense of wonder"]; enough analytic geometry to provide background for calculus; introduction to vectors in form ready for linear algebra.

1. (about one-third) Synthetic Geometry

Adapt the present SMSG axioms, making them fewer by combining; viewpoint: a formulation of the basic properties (listed as incomplete in part). (Not as much time will be needed since they will have had more intuitive experience with geometrical ideas and more experience in proofs.) Treat a sequence of theorems in a systematic way. (No deductive three-dimensional synthetic geometry here.)

2. (about three-fifths) Analytic Geometry

This might be formulated as modelling the synthetic geometry with algebra and real numbers. Do the conventional topics, including angles. Do a lot with graphing functions. Also some on tangents ("slopes," "rate of change," etc.) to non-linear curves as an intuitive background for later work in calculus. (Good place for physical examples; other examples ?) Reference: rates of change in Book T, School Math Project. Do some right triangle trigonometry as needed for treatment of slopes, angles, etc.

Do some with vectors in the plane (modified Moredock, Sandmann). Also curves in parametric form.

Three-space analytic geometry, planes surfaces: graphs of curves (parametric form). Brief look at some geometric problems in 3-space (sphere through four points, etc.).

Discussion of other ways to specify location of a point in plane or space, e.g., polar coordinates, bi-angle coordinates, spherical coordinates (astronomy), etc. No work on equations in polar coordinates.

3. (two-seventeenths) Projective geometry or conic sections or rigorous revisit to synthetic geometry or topology

Report of the Committee on Linear Algebra

Our recommendations:

Axiomatic Synthetic Geometry	1st semester 10th grade
Integrated Analytic Geometry Trigonometry, Linear Algebra	2nd semester 10th grade

Details of above

1. Review coordinatization of plane with addition of coordinates in space
2. Vectors in two and three dimensions
3. Vector sums, scalar multiplication, inner products (with applications)
4. Trigonometry (circular functions)
5. Graphing trig functions
6. Linear transformations and matrices (with applications)
7. Rotations and addition formulas for trig functions
8. Matrix multiplication (with applications, e.g., scheduling, Markov processes, two-person games)
9. Complex numbers in trig form and matrix form
10. Polynomial functions and curve plotting
11. Theory of equations
12. Approximation of roots (flow charting)
13. Conic sections (synthetic treatment also)
14. Tangents to conics, which implies tangents to polynomials must be done first
15. Max-min applications
16. Log and exponential functions

Report of the Committee on Calculus

We agreed that a full-year course should be offered as an option and that some int· tive material should be interwoven with geometry, linear algebra, and probability, where appropriate, in the tenth, eleventh, and twelfth grades, as follows:

1. Limits, using polynomial and rational functions

2. Derivatives of polynomials

3. Applications to rates, tangents, optima with polynomials

4. Area of trapezoid, approximate integration

5. Definite integrals of polynomials

6. Derivatives of exponential function and sinusoids (Assume existence and values of limits needed.)

7. Applications, using exponents and sinusoids in addition to polynomials

For twelfth grade we suggest a full-year course as an option. Intuitive mainly, but some epsilonics. Treatment similar to present, but greater use of practical problems, both for motivation and as exercises.

Report of the Committee on Architecture

We recommend that the materials in grades 7-9 be presented in some natural mathematical order rather than as a series of unarticulated units in arithmetic, algebra, geometry, etc.

We recommend that the tenth grade course be constructed in the same spirit as the courses for grades 7-9. A substantial number of students will take this fourth year of secondary school mathematics, and for many of them it will be their last year. The choice of topics for this year should reflect this fact and should, as far as possible, include the topics which are most important for those students who will not continue in mathematics. It should, however, be made as interesting as possible, in the hopes of enticing students to continue.

We urge those who plan to implement the report of this conference to keep in mind the following maxims:

Maxim 1: The intuitive and the deductive should be carefully blended.

Maxim 2: Devote much thought and care to the construction of problem sets.

It has been pointed out, for example, that the problem sets in First Course in Algebra are designed, most of the time, to force students to think about the material they have just read. Some students find this constant intellectual challenge an overwhelming burden. On the other hand, problems designed only to develop skills may provide a temptation to avoid thought altogether. It is difficult to strike a proper balance.

Maxim 3: Encourage teachers to involve all of their students.

We do not know how this can be done, but we feel that this is a very important point.

Maxim 4: Include some significant, deep applications.

Maxim 5: Include some problems that require thought and analysis on the part of students and teacher jointly.

Maxim 6: Do not write materials so tightly that there are no overlaps or reminders. Build in some deliberate redundancy.

Maxim 7: Don't be pedantic. Include only as much hair-splitting as is appropriate for the audience.

Maxim 8: Remember that the material for grades 7-9 is intended for all students. More than one presentation will be needed.

A presentation for average students probably will not be suitable for the lowest quartile, and for these students modifications in language, order of presentation, etc., must be made. Similarly, material for average students may not be sufficient for better students and better schools. This should be kept in mind as detailed outlines are prepared.

Miscellaneous Notes

In the discussions of the preceding reports, a number of separate comments were made which it was agreed should be recorded.

In connection with Maxim 6 in the Report of the Committee on Architecture, it was pointed out that this is in good agreement with a basic philosophy followed in the first round of SMSG writing. This is that a spiral approach to mathematical concepts should be used, each concept being revisited many times, each time at a deeper and broader level.

It was suggested that there should be occasional problems which are understandable and on which the student can do some calculations but for which he might never see a solution. Example: Given four points in the plane, find another point P such that the sum of the distances from P to each of the four points is a minimum.

It was urged that many problems be provided which contain redundant or unnecessary information. Also, many problems should be provided which contain insufficient information.

It was suggested that consideration be given to a much more intuitive approach to the exponential and logarithmic functions. A formal and complete treatment of these functions had the advantage of introducing ideas, such as area under a curve, which will come up again in calculus. However, this may not be worth the effort and time required to do a careful treatment of these two functions at their first introduction, especially since there will be a second and complete treatment of them in the calculus course.

It was suggested that since the level of formality and rigor will be different in different parts of the sequence, this fact be called to the attention of students. They should be informed whenever the ground rules are changed.

It should be noted that conflicting recommendations on the grade placement and the quantity of formal synthetic geometry appear in various of the preceding reports. It was agreed not to try to reconcile these differences at this conference. It was agreed that a substantial treatment of formal synthetic geometry should appear somewhere near the middle of the six-year sequence. In the discussion some thought was given to the possibility of postponing this to the twelfth grade, but this was eventually rejected. It was pointed out that a substantial number of the leading mathematicians in this country were attracted to mathematics by their high school geometry course. While SMSG has never tried to increase the proportion of pure mathematicians in the population, it would not be wise to do anything which might decrease this proportion.

It was reported that by the end of 1966 a considerable amount of information would be available from the National Longitudinal Study of Mathematical Abilities with respect to the effects of various tenth-grade geometry courses.

Finally, it was recommended that the attention of each reader of this report be called to the article "Goals for Mathematics Instruction" by R. C. Buck, which appeared in the November, 1965, issue of the *American Mathematical Monthly*.

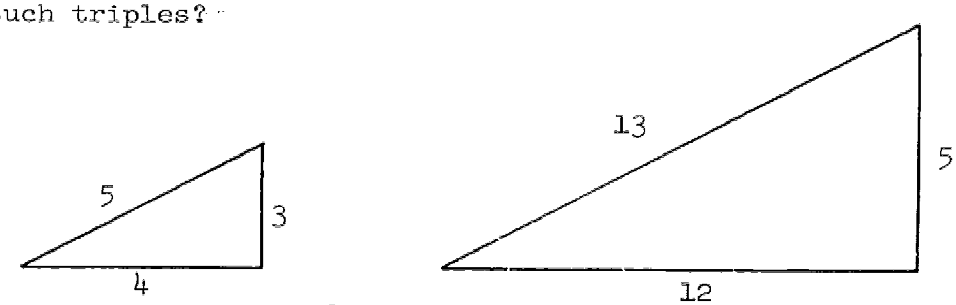
Supplement No. 1

Topic for Inclusion in Analytic Geometry
(related to Number Theory)

R. C. Buck

Starting Point:

Let us find some Pythagorean triples. Sample $(3,4,5)$ since $9 + 16 = 25$. $(5,12,13)$ since $25 + 144 = 169$. Can we find all such triples?



Note that since $(3)^2 + (4)^2 = (\ )^2$, then

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1 .$$

More generally, if $a^2 + b^2 = c^2$, then

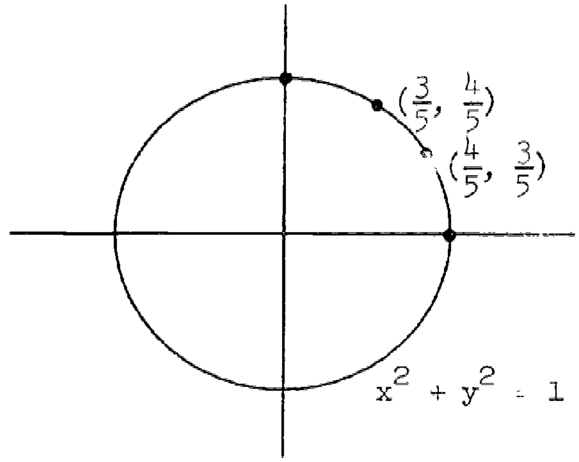
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 .$$

Problem: What are rational numbers x, y with $x^2 + y^2 = 1$?

Re-stated: What are the rational points on the unit circle? i.e., these points $P = (A,B)$ with

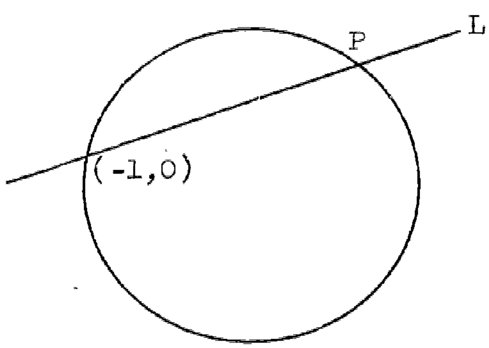
$$A^2 + B^2 = 1$$

and A and B are both rational numbers?



We know a few of these points. Can we find all of them?

<u>Solution</u>: Draw a line L of slope r through (-1,0):



Equation

$$y = r(x + 1) .$$

Suppose L cuts the circle in a second point P = (A,B) then,

$$r = \frac{B - 0}{A + 1} = \frac{B}{A + 1} .$$

Clearly, if both coordinates A and B of P are rational numbers, then r is a rational number. We conclude that every "rational" point P on the circle corresponds to a rational choice of the slope r . (Query: Is the converse true? Is P a rational point if r is rational?)

Let us procede to solve for P , the intersection of the circle and the line L

$$\begin{cases} x^2 + y^2 = 1 \\ y = r(x + 1) \end{cases}$$

$$x^2 + r^2(x + 1)^2 = 1$$

$$(1 + r^2)x^2 + 2r^2x + r^2 - 1 = 0 .$$

We can solve this quadratic equation either by the quadratic formula, or by factoring.

$$x = \frac{-2r^2 \pm \sqrt{4r^4 - 4(r^2+1)(r^2-1)}}{2(r^2+1)}$$

$$= \frac{-2r^2 \pm \sqrt{4r^4 - 4(r^4-1)}}{2(r^2+1)}$$

$$= \frac{-2r^2 \pm \sqrt{4}}{2(r^2+1)} = \frac{-r^2 \pm 1}{r^2+1}$$

$$= -1 \quad , \quad \frac{1-r^2}{1+r^2} .$$

Thus, the coordinates of P , the other point of intersection, is

$$x = \frac{1-r^2}{1+r^2} \qquad y = r(x+1) = \frac{2r}{1+r^2}$$

(This answers the earlier question: if r is a rational number, then the point P is always a "rational" point -- one whose coordinates are both rational.)

We have now solved the original problem. Every rational point on the circle $x^2 + y^2 = 1$ is obtained as follows: choose a rational number r , and then set

$$P = \left(\frac{1-r^2}{1+r^2} \ , \ \frac{2r}{1+r^2}\right) .$$

Take $r = \frac{m}{n}$. Then,

$$\frac{1-r^2}{1+r^2} = \frac{1 - \frac{m^2}{n^2}}{1 + \frac{m^2}{n^2}} = \frac{n^2 - m^2}{n^2 + m^2}$$

$$\frac{2r}{1+r^2} = \frac{2(\frac{m}{n})}{1 + \frac{m^2}{n^2}} = \frac{2mn}{n^2 + m^2} .$$

Hence, the following formula gives <u>every</u> Pythagorean triple

$$(n^2 - m^2 \ , \ 2mn \ , \ n^2 + m^2)$$

where m and n are any integers.

(<u>Check</u>:

$$\begin{aligned}(n^2 - m^2)^2 + (2mn)^2 &= n^4 - 2n^2m^2 + m^4 + 4m^2n^2 \\ &= n^4 + 2n^2m^2 + m^4 \\ &= (n^2 + m^2)^2\end{aligned}$$

TABLE

m	n	$n^2 - m^2$	$2mn$	$m^2 + n^2$
1	2	3	4	5
1	3	8	6	10
2	3	5	12	13
1	4	15	8	17
2	4	12	16	20
3	4	7	24	25
1	5	24	10	26
2	5	21	20	29
3	5	16	30	34
4	5	9	40	41

To check the last triple,

$$\begin{aligned}(9)^2 + (40)^2 = 81 + 1600 &= 1681 \\ &= (41)^2 \ .\end{aligned}$$

<u>Further</u> <u>Problems</u>:

Some circles do not have <u>any</u> "rational" points on them -- e.g.,

$$x^2 + y^2 = \sqrt{2} \ .$$

Some circles have infinitely many -- e.g.,

$$x^2 + y^2 = 1$$
$$x^2 + y^2 = 4 .$$

Are there circles

$$x^2 + y^2 = R$$

which have a finite $(\neq 0)$ number of rational points on them? How does this situation change as R changes?

Partial Solution:

The circle

$$x^2 + y^2 = 2$$

has infinitely many rational points on it.

If one follows the same technique, starting with a different point on the circle, one gets the general formula

$$x = \frac{2mn + n^2 - m^2}{m^2 + n^2} , \quad y = \frac{2mn + m^2 - n^2}{m^2 + n^2}$$

for any choice of integers m and n . For example, we get rational points such as $(1,1)$, $(\frac{1}{5}, \frac{7}{5})$, $(\frac{7}{13}, \frac{17}{13})$, etc.

The circle

$$x^2 + y^2 = 3$$

has no rational points on it(!)

Proof: Let $x = \frac{a}{c}$, $y = \frac{b}{c}$ where the a , b , c are integers, and the fractions are in lowest terms: then, we have

$$a^2 + b^2 = 3c^2 .$$

If we can use congruence mod 3 , then the rest is easy. First, suppose a and b are both multiples of 3. Then, say $a = 3a_o$, $b = 3b_o$. Hence, $a^2 + b^2 = 9(a_o^2 + b_o^2) = 3c^2$ and $3(a_o^2 + b_o^2) = c^2$, so 3 divides c^2 and hence 3 divides c. This was ruled out by the assumption that the fractions were in lowest terms.

Now, observe that if $a \not\equiv 0 \pmod 3$ then $a \equiv 1$ or $a \equiv 2$, and $a^2 \equiv 1$. Since we have to have

$$a^2 + b^2 \equiv 0 \qquad \pmod 3$$

we may experiment with various choices, getting

$$a^2 + b^2 \equiv 1 + 0 \equiv 1 \not\equiv 0$$
$$a^2 + b^2 \equiv 0 + 1 \equiv 1 \not\equiv 0$$
$$a^2 + b^2 \equiv 1 + 1 \equiv 2 \not\equiv 0$$

Hence, no solution exists, and therefore no rational point exists on the circle $x^2 + y^2 = 3$.

Theorem: If $x^2 + y^2 = R$ with R rational, and there is at least one rational point on the circle, then there are infinitely many rational points.

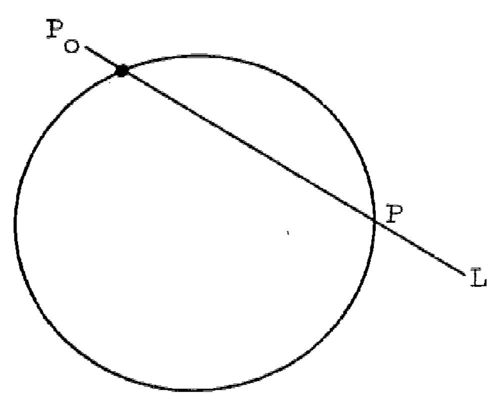
Proof: Let $P_o = (a_o, b_o)$ be a rational point on the circle

$$x^2 + y^2 = R .$$

Consider the line

$$y = b_o + r(x - a_o) = rx + (b_o - ra_o)$$

If there were a second rational point P on the circle, then the slope r of L is rational. We shall prove the converses. Any line L with r rational cuts the circle in a rational point, and thus there are as many rational points as there are such lines, and hence infinitely many.



Solving for the point P of intersection, we have

$$x^2 + (rx + b_o - ra_o)^2 = R$$

$$(1 + r^2)x^2 + 2r(b_o - ra_o)x + (b_o - ra_o)^2 - R = 0 .$$

Since one root of this equation must be $x = a_o$, one factor must be $(x - a_o)$. Since the equation can be written as

$$x^2 + \frac{2r(b_o - ra_o)}{1 + r^2} x + \frac{(b_o - ra_o)^2 - R}{1 + r^2} = 0 .$$

we find the other solution to be

$$x_1 = \frac{(b_o - ra_o)^2 - R}{1 + r^2} .$$

The coordinates of P are then given by this choice of x, coupled with

$$y_1 = b_o + r(x - a_o) .$$

If r is chosen as rational, then these formulas for x_1 and y_1 yield rational numbers. (Recall that a_o and b_o are both rational.)

Further Problems:

Examine other curves for rational points -- e.g.

$$2x^2 + 3y^2 = 1$$
$$x^2 - 2y^2 = 1$$

etc.

What about sums of three squares that add to a square?

Example: $(2)^2 + (3)^2 + (6)^2 = (7)^2$

Solution: Look for rational points $P = (x,y,z)$ on the sphere

$$x^2 + y^2 + z^2 = 1 .$$

Solution: The most general such points are

$$x = \frac{k^2 - m^2 - n^2}{k^2 + m^2 + n^2}, \quad y = \frac{2km}{k^2 + m^2 + n^2}, \quad z = \frac{2kn}{k^2 + m^2 + n^2}.$$

This in turn leads to solutions of the "sum of three squares" problem -- e.g.

$$(8)^2 + (11)^2 + (16)^2 = (21)^2$$

$$(3)^2 + (16)^2 + (24)^2 = (29)^2$$

Footnote: There are, of course, other ways to attack these problems, for example by means of Gaussian primes, etc., but the method above is more elementary and borrows little from the theory of numbers. It also has the advantage of being easy to visualize.

Supplement No. 2

Vectors

R. C. Buck

There are a number of different approaches used in presenting vectors at college or pre-college level. There is a fundamental problem connected with "free" and "bound" vectors, and their role in geometry. The approach suggested below is one which seems quite feasible, and which should make it easier to present this topic at an early (10th grade - 11th grade) level.

The mathematical viewpoint is that vectors, as a group, acts on 3 space (as a manifold). Since it is transitive, and since there is a unique group element carrying a point P into a point Q, the group multiplication corresponds to an algebraic operation on the manifold.

The informal (intuitive) presentation goes thus: vectors are displacements or trips. They can be thought of as specifying the direction and length of a trip (143 miles, NE) etc. One adds displacements by composing the trips.

Observation: on plane, addition is commutative: 10 miles NE + 15 miles SE = 15 miles SE + 10 miles NE.

Same in space. (Harder to see here? How do students react to this?)

After some experience working with simple vector problems (note that displacements act on a geometric <u>origin-free</u> plane, and define a transformation of the entire plane into itself) we can move to a coordinatized form, thus:

Look at coordinate plane (or 3 space) and define addition of points. Then map a displacement V into the point to which it takes the origin: i.e.,

$$V \to V(0)$$

Then observe that

$$V(P) = P + V(0)$$

and that

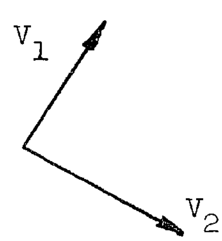
$$V_1 \circ V_2(0) = V_1(0) + V_2(0)$$

and that this gives (an isomorphic) a representation of the group of vectors. Thus, we can do algebra with vectors by working with their coordinate representations.

The difference between this approach, and that that is often taken is that we frankly adopt the view that <u>vectors</u> are entities that <u>act</u> on the geometric plane, and may be combined by succession. Thus, they are <u>functions</u> or <u>mappings</u>. <u>We never add a vector and a point</u>. The final representation of vectors <u>as</u> points enables us to combine (compose) vectors by adding points, and compute the action of a <u>vector</u> V on a point P by adding the points $V(0)$ and P. Thus,

$$V(P) = V(0) + P.$$

The remainder of the treatment can be very classical. Two vectors are $\perp$ if they "act at right angles".



In terms of their representations, this says that $V_1(0) \cdot V_2(0) = 0$. Scalar multiples of vectors have obvious meaning. (The double of a trip V goes twice as far in the same direction.) The coordinate representation yields at once the corresponding formula for scalar multiplication of points.

Connections with <u>forces</u> should be made early. This provides <u>two</u> physical pictures for vectors -- trips and forces. Later on, students will see others -- moments, velocities, accelerations, etc. All of these are <u>physical</u> systems which have vectors as models. The illustrations are easy and the connections vivid. The mathematical formulation of an <u>abstract</u> vector space can wait, but the essential idea that there <u>is</u> a general structure (geometric version) and that the coordinate form is a representation (model) of <u>it</u>, can be put across, I believe.

Note again that this will assist in the algebraic work with negatives. $(-1)V$ is the reverse vector (by definition). But, in the coordinate form $c(x,y,z) = (cx,cy,cz)$. Since the reverse of $(-1,2,-3)$ is $(1,-2,3)$ we must have $(-1)(-1) = 1$, $(-1)(2) = 2$ and $(-1)(-3) = 3$. (Etc.)

Supplement No. 3

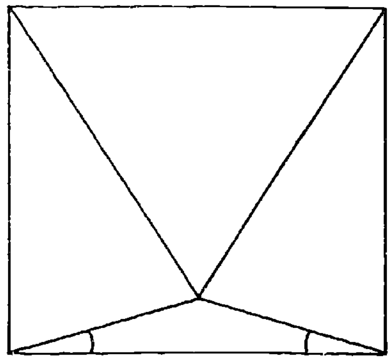
Geometry R. C. Buck

In the treatment of synthetic geometry, I am more anxious that there be adequate experience in problem solving than I am interested in the axiomatics of deduction.

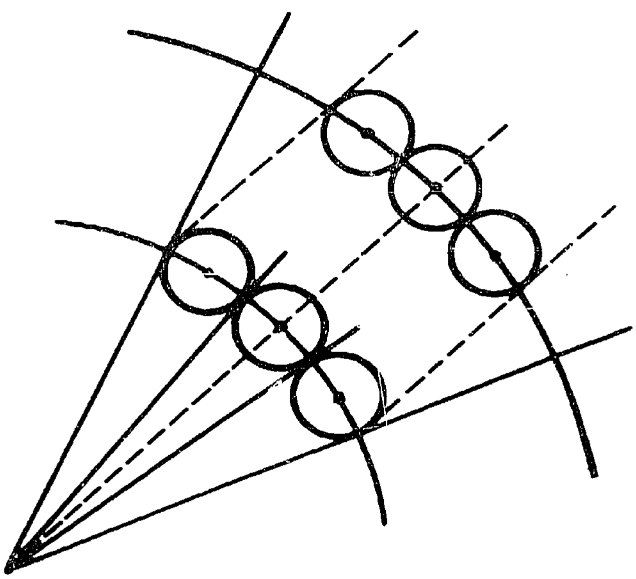
The subject of geometric constructions is, I believe, a good topic for this. It has been argued that the use of ruler and protractor axioms makes most constructions irrelevant and unnecessary. This is certainly true for many -- altho I still feel that it is good for a student to understand and be able to give a convincing argument for the construction of an angle bisector. However, there are many constructions which cannot be done simply by ruler measurement, or protractor measurement. Given three points, draw the circle thru them; given a triangle, inscribe a circle in it; given two points and a line, find a circle thru the points and tangent to the line. With imaginative combinations of protractor-ruler methods, and standard arc swinging techniques, many of these will not be too difficult for the student and will give him rewarding geometric experiences. I strongly urge that quite a lot of these be used as motivation for geometrical work. Note too that one is quite close to the interesting general problem of constructivity; I do not know at the moment if I would recommend any discussion of this in depth at this level (10th grade) but perhaps a clever writer can do something that isn't completely absurd.

More Problems:

Given a square, construct angles of 15° degrees as shown, and then prove that triangle APB is equilateral.



Disprove the following angle trisection. (Everyone has his favorite. Here is an easy one to dispell.)



(1) Draw arc $\widehat{AC}$

(2) Bisect at B

(3) Draw a circle, center at B, and two more of <u>same</u> size, with centers on arc, and touching first.

(4) Draw lines parallel to OB thru the second and third circles.

(5) Draw arc $\widehat{DF}$, mid pt = E

(6) Draw circle, centers at E, of <u>same</u> size as at B, and two more. <u>Result</u> trisects AOC.

*<u>Editor's</u> <u>Note</u>: This is indeed correct. See "Structure of Elementary Algebra", by Vincent Haag, especially Chapter 6, and more especially pages 6.13 to 6.15.

Supplement No. 4

Algebra R. C. Buck

Anyone who takes a serious look at the problem of teaching "traditional algebra", including the solution of rational and algebraic (radical) equations, finds himself faced by several non-trivial decisions. What is a polynomial? Is $2(x + 1)$ a monomial? What about $3(x + y)z$?

Is $$\frac{x^2 - 1}{x - 1} = x + 1 \quad ?$$

Is $$\sqrt{x - 1}\,\sqrt{x + 1} = \sqrt{x^2 - 1} \quad ?$$

Is $$\frac{x^2y^3z^{-5}}{x^{-2}yz^{-6}} = x^4y^2z \quad ?$$

Most texts present a jumble in which no clear policy appears. (This is true also in the SMSG algebra, although one feels that there was conscious choice involved*; indeed, the brunt of my remarks may aim more at recommending a conscious confusion than attempting a complete clarification.)

The first problem arises when we ask if we are to deal with polynomials as functions, or as "forms", or as "expressions".

Let us be specific by explaining exactly what we have in mind about each of these approaches.

(a) Polynomials as functions

Here we are on familiar ground. A polynomial is any of the functions on the set R of real numbers, defined by

$$P(x) = a_0 + a_1x + \ldots + a_mx^m$$

where the coefficients a_j may be arbitrary members of R. The collection of all polynomials forms a ring, and one learns how to do arithmetic with polynomials, i.e., how to add and multiply them, and how to do division (with remainder). Equality of polynomials is equality as functions. A fundamental result is that two polynomials P and Q of degree n which take the same values for more than n distinct choices of input values must take the same values for all input values, and thus be equal. One is also concerned with factorability, and the factor theorem and its consequences are central; a real linear factor $x - c$ divides $P(x)$ if and only if $P(c) = 0$. [Here, "divides" means that there is another polynomial Q with real coefficients such that $(x - c)Q(x) = P(x)$ for <u>all</u> x.]

(b) <u>Polynomials as "forms"</u>

Here, I have in mind the "correct" treatment presented in Van der Waerden (or in W. W. Sawyer's "Concrete approach to Abstract Algebra") in which a polynomial P is viewed as a sequence of numbers ("coefficients")

$$P = (a_0, a_1, a_2, \dots)$$

terminating ultimately in zeros. The entries in the sequence come from a selected field, (or ring). Addition and multiplication are defined by the formal rules

$$(a_0, a_1, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots)$$

$$(a_0, a_1, a_2, \dots)(b_0, b_1, b_2, \dots) = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, \dots).$$

By identification of the polynomial $(a, 0, 0, 0, \dots)$ with the number <u>a</u> and the abbreviation $x = (0, 1, 0, 0, 0, \dots)$ it is then observed that

$$P = (a_0, a_1, a_2, \dots) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots .$$

Polynomials form a ring, and the usual exercises can be given in addition and multiplication.

It is eventually necessary to explain the way in which a polynomial P is associated with a function, so that one is able to speak of the value $P(c)$ for any choice of c (in the coefficient field k). Technically, this correspondence is a homomorphism from the ring of polynomials over k into the ring of functions on k to k. In an adequate treatment, one would give examples of finite fields k in which this mapping is not an isomorphism. (For example, if $k = \{\text{integers mod } 7\}$, then $P = x^7 + 3x^2 - 2x$ and $Q = 3x^2 - x$ are different polynomials, which give rise to exactly the same polynomial function.)

One will also want to discuss factorability, and the factor and remainder theorems.

(c) Polynomials as expressions

In this approach, we start by saying that "expression" shall denote formulas built up by the operation of concatenation, from the following rules:

(i) If α and β are expressions, so is $\alpha\beta$.

(ii) x, +, - are expressions.

(iii) Any numeral is an expression.

Certain simple syntactical rules are then imposed to eliminate such expressions as ++xx - 5x + 7 - , and the smallest resulting class is called the class of "polynomials in x". The formal associative, commutative and distributive laws are imposed, as well as exponents for abbreviation, and two polynomials are regarded as equal only if they can be reduced to identical expressions by application of a "reduction process". (There are bound to be "word problem" difficulties here.)

At some stage, one introduces a procedure by which an expression can denote a function (e.g., Church), but the approach is based mainly upon learning to work with expressions themselves, not functions.

In any one of these three treatments, how do we go on from here? The crucial points are:

A. How do we handle rationals? (I am handicapped by the fact that there is no word that is used for $\frac{x^2 + x - 2}{x - 1}$ which does not already indicate whether this is to be a function, a form or an expression.)

B. How do we handle the case of several variables?

C. What do we do about $\sqrt{}$, $\sqrt[3]{}$, etc?

For example, if polynomials are functions, then $\frac{x^2 + x - 2}{x - 1}$ denotes a function F which is defined for all (real) numbers other than 1. Noting that $x^2 + x - 2 = (x - 1)(x + 2)$, we observe that F is not the same as the polynomial $x + 2$, but that these agree everywhere except at 1. (We are all aware of the problems which these distinctions already cause in an elementary calculus course, where we are able to help by introducing other more challenging examples such as $\frac{\sin x}{x}$, x^x, and $\frac{\log x}{x - 1}$ and discussing limits, continuity, and "removable discontinuities" and extension properties for general functions. Is this approach at all suitable for 9th grade children?)

If polynomials are expressions, then we can introduce $\frac{x^2 + x - 2}{x - 1}$ merely by adjoining an allowed operation of division by the formation of "quotients", and giving arbitrary rules for their use. But, we cannot allow $\frac{x^2 + x - 2}{x - x}$ so that a special arbitrary rule must be made that one is not allowed to form a quotient with the zero expression -- or one equivalent to it -- in the denominator.

If polynomials are forms -- i.e., coefficient sequences --, then one creates rational forms as the field of formal quotients of polynomials. (To do this correctly, one may point out that the ring of polynomials is an integral domain, and then prove in the usual way that any integral domain may be embedded in a field.)

Note that in this approach, we have

$$\frac{x^2 + x - 2}{x - 1} = x + 2$$

with no worry about the denominator. Since $x - 1$ is not zero (the zero polynomial) there is no problem about forming this quotient. The rules for adding quotients arise from the definition, and are not arbitrary but deduced.

What about introducing more than one variable? In the third approach there is no real problem. One merely introduces other primitive expressions such as y, z, u, v, etc., and then the basic syntactical rules yield such expressions as

$$x^2y + uz$$

$$(3x + 5yz)(x^2yz)$$

and, if we have also allowed quotients,

$$\frac{x^2y^3z}{xy^4z^3} + \frac{x + y}{x + z} .$$

In the "correct" second approach, things either become much more complicated, or much more sophisticated. It is possible (but very cumbersome) to define polynomials in several variables as "coefficient arrays" in the form of k dimensional infinite row-finite matrices (multiplication is a horrible [illegible]) A much preferable treatment is to define polynomials ind[illegible]th polynomials in $k + 1$ variables constructed as polynomia[illegible] variable over the ring [illegible]olynomials in k variables. Once this is done, the extension to the field of rational forms is immediate (modulo the fact that these rings of polynomials are integral domains).

Finally, which approach will enable one to speak of $\sqrt{x^2 + x}$ or $\sqrt{x^3y^2}$? If one has adopted the "expression" choice, can one merely enlarge the allowed operations? If so, is there no possible definition for $\sqrt{\ }$ other than the following? "For any expression ϕ . $\sqrt{\phi}$ is an expression, and $\sqrt{\phi}\,\sqrt{\phi} = \phi$." (Which leads at once to the observation that $\sqrt{-\phi}\,\sqrt{-\phi} = -\phi$ and an inconsistency.) As expressions, there is no order relation (unless one decides to refer to any expression such as x^2yz as positive, and $-xy^3z^2$ as negative -- where upon $x^2yz - xy^3z^2$ is presumably neither). In particular, I do not see a way to introduce a definition for $|\phi|$ when ϕ is an arbitrary expression.

If polynomials are forms, then we have somewhat the same problem. Presumably, $\sqrt{P}$ should represent a polynomial whose square is P. However, this will not in general exist, so that we are again faced by an extension problem, and must create entities to solve equations of the form

$$Q^2 = P .$$

Moreover, to have this agree with what we already have, we must agree that $Q^2 = x^2 - 2x + 1$ has only the solutions

$$Q = x - 1$$

$$Q = 1 - x .$$

If, on the other hand, we have adopted the <u>function</u> treatment, then we are forced to say that the equation

$$Q^2 = x^2 - 2x + 1$$

has an <u>infinite</u> number of solutions, one for each set S of real numbers, namely

$$Q(x) = \begin{cases} x - 1 & x \in S \\ 1 - x & x \notin S . \end{cases}$$

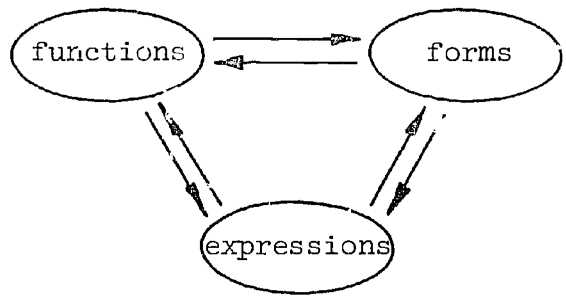
Of these, $Q(x) = \sqrt{x^2 - 2x + 1}$ is only one, namely $Q(x) = |x - 1|$, and <u>this</u> solution isn't even a polynomial!

Here, then, are three different treatments for the elements of algebra. I think it is clear that none is satisfactory in itself for presenting at the high school level (or for that matter at freshman college level!) What is a solution to this problem?

First, it must be pointed out that a practicing mathematician operates simultaneously with all three treatments. In solving an elementary problem dealing with rational functions, he immediately goes over to rational forms, replacing (e.g.)

$$\frac{(x^2 + xy - 2y^2)(x + 7)}{x - y}$$

by $(x + 2y)(x + 7)$. He also handles these as expressions quite happily at that level, feeling free to replace $x + 2y$ by z, etc.



If he were prevented from moving freely between these viewpoints, then his ability to do mathematics would be severly curtailed.

How can we teach elementary algebra so that students acquire the same facility?

The only solution I can imagine is to adopt a form of conscious confusion, blurring certain distinctions, and trying to get across the following viewpoints:

1. Functions are basic, and we want to be able to handle them with as much ease as we do numbers.

2. We have to learn to interpret and work with <u>descriptions</u> of functions. These are usually given in a special way, and so we have to learn to work with the kinds of expressions that we use to describe functions.

3. Expressions for polynomials and for rational functions are more general things than functions, and we can learn to work with them with as much success as we can with numbers. There is always a way to go from an equation between algebraic expressions, to the functions they define, and sometimes there are differences that can arise there -- due to the fact that one has to worry more about things like division by zero.

4. You can think of the whole system as having two levels:

$$\begin{tikzcd}
\boxed{\text{algebraic expressions}} \arrow[d, shift right] \\
\boxed{\text{algebraic functions}} \arrow[u, shift right]
\end{tikzcd}$$

To solve problems down below, you move up to the higher level where the rules are easier to apply, but you do have to examine your results before you come back down again.

5. There are ways to put the whole treatment of rational expressions on a rigorous basic, but they aren't easy to understand on the elementary level. All that it is necessary to understand is that there exists a mathematical system called <u>the field of rational algebraic expressions</u>, (over any given field of numbers?) and that this field contains x , y , z , etc., (called primitive polynomials)(or variables?) and that, like any field, you can add, multiply, divide, etc., and all the usual laws hold. Perhaps the fact that the basic mapping

$$\text{expressions} \longrightarrow \text{functions}$$

is a homomorphism can be put across.

6. By the time this has been done, perhaps a similar treatment can be done for radicals, etc., pointing out that this is <u>far</u> more complex, and that indeed, the system doesn't really work nicely!

Supplement No. 5

"Open Sentences" R. C. Buck

My remarks here will be closely related to those in Supplement No. 4 on algebra. Indeed, as I see it, the parallel is very close. We want students to be able to work with statements of a mathematical nature. Many of these deal with relations between numbers, functions, sets, etc., and involve descriptions of numbers, functions, sets, etc. "The number which has the property that its double is three more than it is." "The rational numbers that are doubled if you increase their numerator and denominator each by 5." [This looks like a meaningful problem, but has a very controversial answer!]

We have built up, over the years, a useful technique of combining English words, letters and special symbols into a relatively unambiguous language for talking mathematics. This language contains expressions such as

$$x < 7$$

$$x^2 - 1 = (x - 1)(x + 1)$$

$$xy = yx$$

The set of all u with $u > 4$.

Each of these is only meaningful in terms of an assigned context. As part of this context, there are certain sets which are the domain of the "free variables" x , y , u , etc. (Note that $>$, $=$, 1 are also variables in a sense, since these do not have context-free meaning.) In terms of the assigned context, an expression such as $x < 7$ can be regarded as a function on the set (of numbers) associated with "x" , and for each selection of x , assigning as value a statement such as $4 < 7$, $7 < 7$, $9 < 7$, etc.

Problem: What is a statement?

Because we want students to be able to work with mathematical statements, we want them to learn to work with the sort of expressions that are used, and to understand the differences between such statements as

$$x > 4 \quad \text{and} \quad x < 8$$

and

$$x > 4 \quad \text{and} \quad y < 8$$

and

$$x < 4 \quad \text{and} \quad x > 8$$

One technique that can be applied to such statements is to construct the "solution sets". But, this is not the only thing one does with such statements!

Furthermore, the way in which such statements arise is not always the same. In particular, "x" may be used as a name for a rather specific object about whose existence there is little doubt. (The number Sally is thinking of; the length of that stick; the distance from here to the center of the earth), or for the name of something to which we wish to give tentative existence. (Suppose that x is a number that obeys $2x^2 + 5 \leq x^2 + 4x$.)

The basic operation that one does with statements is <u>inference</u>. In the context of numbers, one wants sutdents to agree that

$$x < 3 \quad \text{implies} \quad x < 4$$

$$x^2 + 1 \leq x \quad \text{implies} \quad x^2 - x \leq -1$$

$$xy = 0 \quad \text{implies} \quad x = 0 \quad \text{or} \quad y = 0 \ .$$

It happens that each of these can be reudced (via "solution sets") to an observation that two sets are related by inclusion. <u>But, you do not want students to infer the implication by verifying the set inclusion!</u> (In computer language, this allows only the technique of exhaustive search.)

The method of exhaustion <u>is</u> a valid technique (exploited, for example, in the "method of undetermined coefficients") but at the early stage we are discussing, it should not be made the goal, but only an alternative.

Thus, while one may wish to utilize "solution set" as an expository device, it should not be made central to the treatment.

Expository Articles on Applications of Mathematics

I. Articles probably at an appropriate level for students in grades 7-10.

1. Alfred, Brother U., "The Importance of Elementary Operations," M. T., (December, 1963), 65:614-619.

   Subject matter: Operations; some simple probability
   Mathematics used: Arithmetic
   Summary: Since most computations require several operations, 95% accuracy in single operations may give a score of 65% on a computation examination. Both the problem and the message would be useful to students.

2. Blass, G. A., "On the 'Clock Paradox' in Relativity Theory," A. M. M., (October, 1960), 67:754-755.

   Subject matter: Relativity; Lorentz transformations
   Mathematics used: Simple algebra
   Summary: Though the complicated aspects of relativity theory are not touched here, this is a simple, formal way of explaining apparent paradoxes of that theory. The Lorentz transformations which do the work are of course taken on faith.

3. Eells, W. C., "One Hundred Eminent Mathematicians," M. T., (November, 1962), 55:582-588.

   Subject matter: History, ranking, probable error
   Mathematics used: none
   Summary: This article is interesting in its subject matter and in its discussion of a method of establishing a ranking and the errors inherent in such procedures. The matter of assumptions made in applications is well illustrated by comparing the assumptions of his method with assumptions used by others in attempting to make such listings.

4. Eves, H., "The Latest About Pi," M. T., (February, 1962), 55:129-130.

   Subject matter: Computation of pi and e, uses of computers
   Mathematics used: none (The two formulas given are in the form of arc tan functions.)
   Summary: An excellent short piece on one use of computers. Professor Eves does not tell us why anyone would want these numbers computed to so many decimal places.

5. Fischer, I., "How Far is it from Here to There?" M. T., (February, 1965), 57:123-130.

Subject matter: indirect measure; mathematical models
Mathematics used: none
Summary: A well-written and useful article. It illustrates very well the uses of abstract models in solving problems. It should be useful to junior high school students, while there are enough indications of unsolved problems to interest even a fairly advanced high school student.

6. Fogo, J. E., "Linear Indeterminant Problems," M. T., (April, 1964), 57:223-225.

Subject matter: As per the title
Mathematics used: Solutions of linear equations, systems of linear equations and linear inequalities
Summary: A well-written, short article that might succeed in getting students to actually think about the problems they are solving instead of relying on mechanical manipulation.

7. Foster, C., and Rapoport, A., "The Case of the Forgetful Burglar," A. M. M., (February, 1958), 65:71-76.

Subject matter: A special sort of random walk in one direction
Mathematics used: Main part--ability to follow a simple proof
Summary: It is a useful demonstration that a problem can arise in several contexts and have several interpretations. The definition-lemma-theorem chain is also a useful, simple example of standard mathematical procedure.

8. Gager, W. A., "Computing With Approximate Data," School Science and Mathematics, (January, 1965), 68-83. (P. Peak, M. T., May, 1965; 58:438)

"Some of the discussion revolves around exact numbers obtained by counting indivisible units, numbers used to express measurements as approximations, the absence of absolute precision, maximum and relative amount of error, and the size of units."

9. Giles, R. T., "Building an Electrical Device for Use in Teaching Logic, M. T., (March, 1962), 55:203-206.

Subject matter: Logical operations
Mathematics used: none
Summary: The circuit given is simple and would be easy to duplicate. Sufficient detail is given. Cost of parts is estimated at $15. Some students would find it an extremely interesting article.

10. Golomb, S. W., "Checkerboards and Polyominoes," A. M. M., (December, 1954), 61:675-682.

Subject matter: Covering a checkerboard with various polyominoe shapes.
Mathematics used: None
Summary: This is, of course, a "puzzle" type problem but it is an unusually good exposition and the proofs that are given make it mathematical rather than "merely" a puzzle. It would be interesting to students from junior high school up.

11. Hamming, R. W., "Intellectual Implications of the Computer Revolution," A. M. M., (January, 1963), 70:4-11.

Subject matter: As per the title
Mathematics used: none
Summary: Computers have improved in speed by at least six orders of magnitude, with a great increase in reliability and a decrease in cost of something more than a thousand times. This is an exceptionally interesting and lucid discussion of the implications of these facts for our lives and for the applications of mathematics. It would be an excellent article for a high school student.

12. Gramann, R. A., "A Queuing Simulation," M. T. (February, 1964), 57:66-72.

Subject matter: Queuing theory, probability, random numbers
Mathematics used: Formulas for the Pousson distribution and computations are shown in tables already done
Summary: The article is an explanation in simple terms of queuing problems. It is well written and very clear and would be interesting to students from, say, the eighth grade on.

13. Hetrick, J. C., "Mathematical Models in Capital Budgeting," Harvard Business Review, (January, 1961), 49-64. (P. Peak, M. T., January, 1962, 55:51)

"The three major factors in most cases are manufacturing, distribution, and sales. The questions and the factors are represented symbolically and graphed to show lines of intersection and planes of operation. By this method management is able to develop patterns of operation so that they may be used as models. All factors can be represented in the model and the composite is the aid to decisions which must be wisely made."

14. Horton, G. W., "A Boolean Switchboard," M. T., (March, 1965), 58: 211-220.

Summary: This is another article explaining the operations of Boolean algebra and relating them to electrical circuits. It gives a circuit diagram and some pictures of a switchboard which incorporates the circuits for the Boolean operations.

15. Kane, R. B., "Linear Programming, an Aid to Decision Making," M. T., (March, 1960), 53:177-179.

Subject matter: Linear programming in two and three dimensions
Mathematics used: Linear equations and inequalities and graphing in the plane and in space
Summary: Two good problems in linear programming accessible to high school students.

16. Landin, N. P., "Mechanics of Orbiting," M. T., (May, 1959), 52:361-364.

Subject matter: Newton's Laws; uniformly accelerated motion, gravitation, centrifugal force, potential and kinetic energy, viewing angle (all with appropriate formulas).
Mathematics used: Principally algebra; one equation requires trigonometry
Summary: An interesting discussion; fairly nontechnical; accessible to any student good at algebraic and arithmetic manipulation.

17. Levin, J., "Variable Matrix Substitution in Algebraic Cryptography," A. M. M., (March, 1958), 65:170-179.

Subject matter: As per title
Mathematics used: Congruences (mod 26), matrices
Summary: The simple example given in the first part of the article could be explained to anyone who has dealt with "clock arithmetic," say, to junior high school youngsters. The more complicated methods require a great deal more mathematics, but might be appropriate in a junior or senior class that has had some linear algebra.

18. Lichtenberg, D., and Zweng, M., "Linear Programming Problems for First Year Algebra," M. T., (March, 1960), 53:171-179.

Subject matter: Linear programming, graphing linear equations, polygonal regions in a plane
Mathematics used: Linear equations and inequalities
Summary: A good source of simple linear programming problems. The article could probably be used as is by students.
(See also: Abigal, S., "Fundamental Theorem of Linear Programming," M. T., (January, 1961), 64:25-26. This article gives a proof of the main linear programming theorem.)

19. Luce, R. E., "The Mathematics Used in Mathematical Psychology," A. M. M., (April, 1964), 71:364-378.

Subject matter: As per the title
Mathematics used: none
Summary: This would be useful to both mathematics teachers and their students as an example of a field that has only recently begun to

use mathematics extensively. There is an extensive bibliography and a listing of "models" used in various fields of mathematical psychology. He illustrates very nicely what is meant by mathematical models.

20. Manheim, J., "Word Problems or Problems with Words," M. T., (April, 1961), 54:234-238.

Subject matter: Word problems
Mathematics used: none
Summary: The author makes the distinction between real word problems, in which one must have a prior knowledge about the relationships governing the words, and imaginary word problems in which one can find out all he needs to know simply from the structure of the problem. The article is sound and well written and would be useful either handed out to students or as a guide to a teacher.

21. Martin, E. P., "The Mathematics Market," M. T., (December, 1959), 52:616-618.

Subject matter: Mathematics that students of biology should understand and mathematics as used by professional biologists
Mathematics used: none
Summary: The second half of the article would be excellent reading for most secondary school youngsters.

22. Merrill, C. F., "Some Reflections on Gulliver's Travels," M. T., (December, 1961), 54:620-625.

Summary: The author discusses Swift's satires on mathematics and science and gives a number of examples of his lack of sympathy for pure mathematics and science. A long section discusses the departure from scientific reality in the biological aspects of Swift's writing. (Following this article, the editor, Howard Eves, discusses the surprising coincidence whereby Swift predicted accurately the existence of two satellites of Mars long before their existence was verified.)

23. Mosteller, F., "Understanding the Birthday Problem," M. T., (May, 1962), 55:322-325.

Subject matter: Birthday and birth-mate problems
Mathematics used: A little probability, logarithms for computation, manipulation of algebraic expressions of fairly complicated sorts
Summary: An excellent and lucid discussion. The discussion itself should be interesting to students from, say, junior high on. To follow the derivations of the actual results, advanced algebra would be needed.

24. Ore, O., "An Excursion into Labyrinths," M. T., (May, 1959), 52:367-370.

Subject matter: Graph theory
Mathematics used: Some simple combinatorics
Summary: A discussion of certain graph problems motivated by an introductory discussion of mazes. Not really an application but a pseudo application. Historical allusions make it interesting.

25. Ore, O., "Going Somewhere?" M. T., (March, 1960), 53:180-182.

Subject matter: Graphs and networks
Mathematics used: Algebra and inequalities and sophistication about notation
Summary: Ore states that problems concerning minimum distance or minimum time or minimum expense can be attacked by graphs or networks consisting of edges connecting vertices or branch points. He then sets up a problem and explains an elegant solution to it, but then observes that in most applications it is highly impractical. He then discusses a second method related to his earlier article on labyrinths.

26. Polya, G., "Heuristic Reasoning in the Theory of Numbers," A. M. M., (May, 1959), 66:365-384.

Subject matter: Prime numbers, twin primes, probability, methods of heuristic reasoning
Mathematics used: Only prime numbers in the first half
Summary: This superb article could be used on (at least) two levels. The first half, simply presenting the data from 30 million primes and the results for prime pairs, would be a good exercise in heuristic reasoning for students in seventh grade or beyond. The message of the second half would be useful for almost any student. The entire article would be accessible to a good sophomore or junior student.

27. Polya, G., "The Minimum Fraction of the Popular Vote That Can Elect the President of the U. S.," M. T., (March, 1961), 54:131-133.

Subject matter: Given by title
Mathematics used: Some algebra, inequalities, and some measure (all fairly simple)
Summary: On the basis of two assumptions it computes the minimum number of popular votes that would lead to an electoral college majority. (It turns out that this fraction is just a little over 22%.) The article closes with a discussion of applied problems. Very nice!

28. Raiffa, H., "Normative Decision Models," M. T., (March, 1959), 52:171-179.

Subject matter: Mathematics models for decision making; linear programming
Mathematics used: Arithmetic, basic probability
Summary: Interesting, non-technical, illuminating. Good motivation on the uses of mathematics. Examples of uses given. Accessible to almost any interested student, say, seventh grade or later. Nice, breezy style.

29. Rankel, P. J., "Quantification in the Social Sciences," M. T., (January, 1962), 55:20-33.

Subject matter: Measurement, measure theory, quantification, relations
Mathematics used: none
Summary: This is an excellent discussion of measure for students or for teachers. The author's style is conversational and is extremely clear. I would regard it as useful at any level, say, from a good sixth grade student up.

30. Read, C. B., "An Obsolete Problem in Arithmetic," M. T., (May, 1959), 52:366-367.

Subject matter: Currency conversions
Mathematics used: none
Summary: This is a brief discussion of problems found in American arithmetic books of the early nineteenth century, devoted to conversions between various state currencies. It would be useful at the elementary or junior high level.

31. Read, C. B., "Can Present Day Students Work Problems of Seven or Eight Generations Ago?" M. T., (November, 1963), 56:538-540.

Subject matter: Old problems
Mathematics used: Arithmetic
Summary: The problems are from a notebook of a student who apparently took the problems down from dictation by a teacher. Dates given are from 1784 to 1800. The problems would be interesting to junior high or high school students.

32. Scheid, F., "Clock Arithmetic and Nuclear Energy," M. T., (December, 1959), 52:604-607.

Subject matter: Modular arithmetic, random numbers, simple probability, penetration of nuclear particles, uses of computers
Mathematics used: Simple arithmetic
Summary: The author asks about the probability of a messenger getting through a crowd in Times Square by running through. bouncing off people, and proceeding by such random collisions. The problem is then restated as a problem of nuclear particles penetrating a sheet

of lead. This is a simple, very well-written article of interest to mathematics students from late elementary school on. It is somewhat patronizing perhaps for a high school student.

33. Scheid, F., "Intuition and Fluid Mechanics," M. T., (April, 1960), 53:226-234.

Subject matter: A certain problem in fluid mechanics, f/ = ma.
Mathematics used: Supposedly just arithmetic but actually some sophistication in mathematical symbolism and notation.
Summary: The article is well written and accessible to, say, a good algebra student. Various maneuvers in solving "applied" problems are very nicely laid out. Both the fruitful role of intuition and the dangers of carrying intuition too far are nicely illustrated.

34. Scheid, F., "A Tournament Proble[illegible] A. M. M., (January, 19[illegible]0), 67:39-41.

Subject matter: Arranging bridge and tennis contests under certain restrictions
Mathematics used: A couple of simple formulas
Summary: This would be a good problem, say, for ninth grade or above, to indicate how restrictions on a problem determine its solution. One feels that the problem is of marginal value for instruction, but an interesting puzzle problem.

35. Scheid, F., "Some Packing Problems," A. M. M., (March, 1960), 67:231-235.

Subject matter: Placement of the pieces on a chessboard under certain restrictions
Mathematics used: Inequalities, simple algebraic formulas, some facility in following an argument or figuring out a pattern for oneself
Summary: The problem on a chessboard is elementary and would be an interesting "discovery" problem for any good secondary school youngster. The generalization of the problem requires more sophistication.

36. Segall, M. H., Campbell, D. T., and Herskovits, M. J., "Cultural Differences in the Perception of Geometric Illusions," Science, (February 22, 1963), 769-771. (P. Peak, M. T., October, 1963, 56:460)

"This article is a report on a six-year study of children and adults from Africa, the Philippines, and the people in Evanston, Illinois. Your students will be interested in the charts showing means of primitive tribes compared to others."

37. Stakes, G. D. C., "Linkages for the Trisection of an Angle and Duplication of the Cube," Proceedings of the Edinburgh Mathematical Society, (December, 1960), 1-4. (P. Peak, M. T., April, 1962, 55:250)

"Problems about the trisection of the angle and duplication of the cubes are almost infinite. This is not such an article; it is one which shows the linkage as well as how it achieves its goal. Your student can make these simple linkages and justify their use."

38. Thom, A., "The Geometry of Megalithic Man," Mathematical Gazette, (May, 1961), 83-92. (P. Peak, M. T., April, 1962, 55:248)

"As you read this extremely fascinating article, reflect on what these semicivilized peoples were attempting to develop and how they seemed intentionally to adhere to present mathematical principles."

39. Thumm, W., "Buffon's Needle: Stochastic Determination of Pi," M. T., (November, 1965), 58:601-607.

Subject matter: Pi, Buffon Needle problems with approximations to pi
Mathematics used: The first problem and solution involve simple probability, some approximations from graphs, some simple trigonometry and some inequalities. The other solutions require a considerable amount of mathematics.
Summary: An adaptation of the first solution could probably be used with junior high or algebra or geometry students. To read and understand the entire article would require a student with considerable mathematical maturity.

40. Van Tazel, L. T., "Notes on a 'Spider' Nomograph," M. T., (November, 1959), 52:557-559.

Subject matter: Lens and optics problems; parallel resistance problem; construction of nomograph
Mathematics used: Simple algebra
Summary: The article is poorly written and confusing but does describe an interesting technique.

41. Wilson, R. H., Jr., "The Importance of Mathematics in the Space Age," M. T., (May, 1964), 57:290-297.

Subject matter: Uses of elementary mathematics; f = ma, gravitational attraction, air resistance, 4 p times Ap squared, formulas relevant to torque, spin rate, etc.
Mathematics used: A number of formulas are used for illustration
Summary: The article packs a good deal of information into a few pages. It is well written and would be accessible to a good secondary school youngster. Much mathematics is referred to but not very much would be required to understand the article. The exposition is excellent and is about at the Scientific American level.

II. Articles Probably Appropriate for Students in Grades 10-12

1. Beyers, M., "Are Earth-Measured Values Valid in Space?" Science and Mathematics Weekly, (February 20, 1963) 230-231, and (February 27, 1963), 244-245. (P. Peak, M. T., October, 1963, 56:448)

   Summary: "Reviewing the records we find certain 'constants' yielding different values at different times of the year or in different locations. Newton and Einstein are being questioned. Some say the gravitational constants need correcting; some, the force of attraction of two bodies. New explanations of gravity are being considered."

2. Bleicher, M. N., and Toth, L. F., "Circle-Packings and Circle-Coverings on a Cylinder," The Michigan Mathematical Journal, (December, 1964), 337-341. (P. Peak, M. T., May, 1965, 58:437)

   Summary: "The circle-packing problems have always been very interesting, both for their intrinsic beauty and their applications in fields of applied mathematics. This article deals with the problem of packing and cylinder covering."

3. Bochner, S., "The Role of Mathematics in the Rise of Mechanics," American Scientist, (June, 1962), 294-311. (P. Peak, M. T., March, 1963, 56:145)

   Summary: "This interesting article gives a clear and concise picture of the changes that have taken place in the relationship which has existed between mathematics and mechanics since before the seventeenth century. The seventeenth century was an age of revelation; the eighteenth, an age of patriotic organization; the nineteenth, an age of canonical legislation; while the twentieth is an age of reformation and counter reformation."

4. Burington, R. S., "On the Nature of Applied Mathematics," A. M. M., (April, 1949), 56:221-242.

   Subject matter: As per the title
   Mathematics used: none
   Summary: This is an excellent introduction to what is involved in applied mathematics and would be extremely useful for both high school teachers and their students. The summaries of the various fields in which applied mathematics is used are very useful, particularly since specific examples are given. This material may be dated by now and one could wish for an updated version.

5. Clifford, E. L., "An Application of the Law of Sines; How Far Must You Lead a Bird to Shoot it on the Wing," M. T., (May, 1961), 54:346-350.

Subject matter: As per the title
Mathematics used: Trigonometry, the law of sines, and some algebraic manipulations
Summary: The first four pages of the article would be a suitable exposition for a student and would shed some light on the necessity for simplifying assumptions, etc.

6. Dantzig, G. B., "New Mathematical Methods in the Life Sciences," A. M. M., (January, 1964), 71:4-15.

Subject matter: As per the title
Mathematics used: Rather complicated equations, summations, log function, linear equations
Summary: The first and last pages give an interesting discussion of what the prospects are for applications of mathematics to biology. The middle pages detail a model that is interesting to scan but would be inaccessible in its details to most high school students and most mathematics teachers.

7. Davies, H. K., "Practical Experimentation in the Teaching of Basic Statistics," The Mathematical Gazette, (October, 1965), 271-280. (P. Peak, M. T., May, 1965, 58:437)

Summary: "Her thesis is that the student must experience some of the frustration with his own data before he can become fascinated with the statistics. She makes use of simple computing machines, dice, probability tables, and group experiments. Biology studies with animals and plants are used."

8. deBethune, A. J., "Child Spacing; the Mathematical Probabilities," Science, (December 27, 1963), 1629-1634. (P. Peak, M. T., May, 1964, 57:354)

Summary: "As is true in many studies in the field of sociology, an assumption must be made or basic factors must be established. After the establishment of the security factor, the author does a very nice development of the mathematics related to it. The simulated experiments are excellent illustrations of changing probabilities."

9. Drenick, R. F., "Random Processes in Control and Communication," Science, (September 30, 1960), 865-870. (P. Peak, M. T., March, 1961, 54:137)

Summary: "What changes have taken place in the applications of mathematics in the past fifteen years? Read this article for one interesting application to the 'signal space.'"

10. Gale, D., [illegible] Shapley, L. S., "College Admissions and the Stability of Marriage," A. M. M., (January, 1962), 9-15.

Subject matter: A demonstration of how mathematics is applied
Mathematics used: None. Two theorems are proved.
Summary: The authors demonstrate without the use of any formal mathematical apparatus a procedure which any mathematician will recognize as being mathematical. This should be an exceptionally useful article to use with a lay audience or with students at any secondary level, especially perhaps at the later secondary levels where college admission is a preoccupation. Very clear exposition.

11. Gleason, A. M., "Evolution of an Active Mathematical Theory," Science, (July 31, 1964), 451-457. (P. Peak, M. T., February, 1965, 58:139)

Summary: "The author presents an interesting problem by starting with a Police Department sign erected in the street and extends this through differential equations, integration, and generalization to space and planetary motion. This interesting article culminates in a discussion of a very new subject, 'Differential Topology.'"

12. Isaaks, R., "Optimal Horse Race Bets," A. M. M., (May, 1953), 60:310-315.

Subject matter: Optimal betting procedure
Mathematics used: Some calculus, limits, etc.
Summary: The derivations are clearly too complicated to be followed by high school students. The suggestion that exact solutions exist to such problems would be of interest, and the fact that such a seemingly trivial problem is related to some more serious mathematical problems makes a useful point.

13. Johnson, S. M., "A Tournament Problem," A. M. M., (May, 1959), 66:387-389.

Subject matter: Tournament rankings, inductive devinition, recursive techniques
Mathematics used: Mostly skill in manipulationg notation; some maturity
Summary: This article might well be used with advanced high school students to show that problems that are easily stated sometimes have rather complicated solutions and to demonstrate one such solution.

14. Kennedy, E. S., and Hamaeanizadeh, K., "Applied Mathematics in Eleventh Century Iran: Abū Ja' Far's Determination of the Solar Parameters," M. T., (May, 1965), 58:441-446.

Summary: This is another article giving a solution to an ancient problem in the second half of the eleventh century in Iran. It is said that this may be the earliest existing piece of mathematical writing in the Persian language. A translation of the solution of the problem is given along with a detailed line-by-line commentary. Solution is given. Students may well find it interesting to untangle the problem as solved some hundreds of years ago.

15. Kennedy, E. S., and Haydar, S., "Two Medieval Methods for Determining the Obliquity of the Ecliptic," M. T., (April, 1962), 55:286-290.

Subject matter: Astronomy, as per the title
Mathematics used: Trigonometry of a fairly complicated sort
Summary: Interesting chiefly as an exercise in untangling an ancient computation of a "practical" problem.

16. Klamkin, M. S., "A Moving Boundary Filtration Problem, or the 'Cigarette Problem,'" A. M. M., (December, 1957), 64:710-715.

Subject matter: Filtration through a burning cigarette
Mathematics used: Calculus
Summary: Useful principally to indicate the range of problems studied by mathematicians. A middle section summarizing results is accessible, but the remainder of the article requires too fancy mathematical tools to be followed by most high school students.

17. Lambek, J., "The Mathematics of Sentence Structure," A. M. M. (March, 1958), 65:154-170.

Subject matter: As per the title
Mathematics used: Logical structure (logic ?)
Summary: Interesting as an indication of the variety of things which a mathematician can concern himself with. It is obviously only an introduction to a complicated field. One wonders what tie-in this has with the so-called "new English" that is said to be coming into prominence in high schools.

18. Langlois, W. E., "The Number of Possible Auctions at Bridge," A. M. M., (August-September, 1962), 69:634-635.

Subject matter: Combinatorial problems in bridge
Mathematics used: Combination and summation symbols; otherwise, fairly simple
Summary: To a student who plays bridge this should be a very interisting article. Others may be interested in the results: about 6.3×10^{11} possible deals, 5.4×10^{28} possible hands, and 1.3×10^{47} possible auctions.

19. Mudzi, J. S., "Probability and the Radioactive Disintegration Process," M. T., (December, 1961), 54:606-608.

Subject matter: As per the title
Mathematics used: Some probability, much algebraic manipulation, limits
Summary: This is a very concise exposition of the subject matter given by the title. It would be suitable for students who have had some elementary functions and probability.

20. Mulholland, H. T., and Smith, C. A. B., "An Inequality Arising in Genetical Theory," A. M. M., (October, 1959), 56:673-680.

Subject matter: Evolutionary genetics, inheritance of blood type
Mathematics used: In the first half of the article, not a great deal. In the second half a considerable amount, including partial differentiation, inequalities, and complicated summations.
Summary: The first half of the article might well be adapted by a knowledgeable teacher as an illustration of applied mathematics.

21. Nadir, N., "Abū-al - Wafā' on the Solar Altitude," M. T., (October, 1960), 53:460-463.

Subject matter: Telling time from the solar altitude
Mathematics used: Trigonometry and some mathematical maturity
Summary: This is primarily of historical interest, but a good student might well find it interesting to untangle the ancient methods and the translation of them into modern methods.

22. Read, R. C., "Card-guessing With Information--A Problem in Probability," A. M. M.

Subject matter: As per the title
Mathematics used: Probability formulas and advanced algebra
Summary: An interesting problem for students studying probability at the junior or senior level.

23. Rosentberg, H., "Modern Applications of Exponential and Logarithmic Functions," School Science and Mathematics, (February, 1960), 131-138.

24. Schild, A., "The Clock Paradox in the Relativity Theory," A. M. M., (January, 1959), 66:1-18.

Subject matter: Relativity theory, Minkowski geometry, various transformations, Lorentz transformations
Mathematics used: Some calculus. There is a discussion of several kinds of geometry.
Summary: Although a nice exposition that is easy to follow, the mathematical development is probably beyond most high school students. An excellent junior or senior might well read the article with profit.

25. Seimens, E. F., "The Mathematics of the Honeycomb," M. T., (April, 1965), 58:334-337.

Subject matter: As per the title
Mathematics used: Geometry, some algebra and graphing. The use of calculus and trigonometry for a precise solution is indicated, but is not gone into in detail.
Summary: A very interesting little article, usable at tenth grade and beyond. The footnotes are extensive enough to guide an interested student to a complete analysis of the problem.

26. Stoneham, R. G., "A Study of 60,000 Digits of the Transcendental 'e,'" A. M. M., (May, 1965), 72:483-500.

Subject matter: As per the title
Mathematics used: Quite a lot of the mathematics of statistics plus a number of tables and graphs
Summary: Few high school students could follow the derivations that are given, but the discussion of the problem itself, the reasons for interest in it, and the methods used in studying it would be of interest. It is interesting to note that the different digits do not appear to have the same form of distribution, though with the possible exception of 6, the frequency of appearance is the same.

27. Stober, D. W., "Projectiles," M. T., (May, 1964), 57:317-322.

Subject matter: The path of projectiles
Mathematics used: Vectors, trigonometry, considerable algebraic manipulation, some maturity in following algebraic derivations
Summary: The mathematical discussion could be followed only by a student who had completed a substantial unit in trigonometry. A very simple projectile throwing and sighting device is described that could be used at several levels.

28. Stretton, W. C., "The Velocity of Escape," M. T., (October, 1963), 56:400-402.

Subject matter: Escape velocity for artificial satellites; formulas for work, attraction of gravity, and kinetic energy
Mathematics used: Calculus at a fairly simple level, considerable manipulation of formulas
Summary: This is a good enough exposition of an interesting, current problem. The mathematics used are fairly advanced, but a good junior or senior could follow it.

29. Teller, E., "The Geometry of Space and Time," M. T., (November, 1961), 54:505-514.

Subject matter: Theory of relativity
Mathematics used: Analytic geometry, algebra, some sophistication in dealing with formulas and manipulation of them.
Summary: Since this is an unedited talk, it is not suitable as it stands as an exposition for students. It does discuss concepts of relativity in a fairly clear and useful way.

30. Thompson, R. A., "Using High School Algebra and Geometry in Doppler Satellite Tracking," M. T., (April, 1965), 58:290-295.

Subject matter: Doppler effect, rate of change, velocity, range
Mathematics used: Some simple calculus; mainly analysis of graphs via algebra and geometry
Summary: Too much is covered in too short an article. The considerable information that can be obtained by graphical methods is indicated, but for the most part the article is too sketchy and would require supplementing.

31. Toth, L. F., "What the Bees Know and What They Do Not Know." Bulletin of the American Mathematical Society, (July, 1964), 468-481. (P. Peak, M. T., May, 1965, 58:440)

Summary: "...The author proceeds to show us where the bees do well and where they do not...The rhombic dodecahedron gives the best ratio of volume to surface while also acting as space fillers, but there is a better one, namely, the truncated octahedron. Now if the bees would stop at an optimal height as they build from bottom to top they then would have a better system, that is, the bisected rhombic dodecahedron...The author extends this to other illustrations with drawings and mathematical relations."

32. Walkup, D. W., "Covering a Rectangle with T-Tetrominoes," A. M. M., (November, 1965), 72:986-988.

Summary: As with similar articles, this deals with covering a grid with certain tetrominoes. Only elementary mathematics is used.

33. Bleicher, M. N., and Toth, L. F., "Two-Dimensional Honeycombs," A. M. M., (November, 1965), 72:969-973.

Subject matter: Isoperimetric problems; a specialization of the honeycomb problem
Mathematics used: Mostly geometry; one minimum is found via a derivative.
Summary: An isoperimetric problem is defined, and an analogous two-dimensional problem is dealt with by defining a "comb" in a certain way, then by way of fairly simple geometric arguments producing a number of diagrams and proving two theorems. The article would be usable with, or prior to, the several more elaborate expositions on the honeycomb. Except for the one derivative, the mathematics could be handled by a good geometry student.

III. General or Unclassified Articles with Respect to Mathematical Models or Applications

Note: 1) Most of the articles in this list are expository and express a certain general point of view. In most cases the title describes the contents and little mathematical apparatus is used. Hence the "Subject matter" and "Mathematics used" categories used in previous lists have been dropped. 2. The articles in all three lists were gleaned mainly from The Mathematics Teacher or The American Mathematical Monthly of the past decade. Other sources could be used to extend the lists.

1. Baum, John D., "A Second Report on the Teaching of Mechanics in Schools" (a report prepared for the Mathematical Association, London, 1965), A. M. M., (January, 1966), 73:78.

   Summary: "Besides detailed course descriptions and a brief history of mechanics up to the time of Newton, the report presents a philosophy of teaching of mechanics in school. The philosophy is best stated in the words of the report: 'We have no hesitation in commending mechanics as a part of a secondary school mathematics course, whatever the age or grade of the pupils and whether they be in the early years in a secondary modern school or preparing for university entrance. It is a subject which, more than most, can give reality to mathematics by showing how it may be applied. In the hands of a skilled teacher it can be used to stimulate pupils to face and overcome the mathematical difficulties that present themselves.'"

2. Allendoerfer, C. B., "The Narrow Mathematician," A. M. M., (June-July, 1962), 69:461-469.

   Summary: In speaking of applications he says, "Insofar as possible the reform movement has tried to replace applications of this kind i.e., unrealistic and obsolete with realistic applications to matters of current importance. This is perhaps the most difficult aspect of our problem and much more work needs to be done. Let us be as imaginative about these matters as we have been about other areas of the movement." He takes a pot shot at critics of the Morris Kline variety and speaks in closing of "new customers of mathematics" and what can be done for them. This will be useful for teachers. His description and diagram of how a mature mathematical theory is structured and its origins as well as his comments on how new mathematical theories develop could be usefully adapted for use with high school students, but the same ground is covered better elsewhere.

3. Bell, G., and Sons, Ltd., "Applications of Elementary Mathematics" (a compendium prepared for the Mathematical Association, London), 1964.

Summary: This is a 38-page pamphlet with brief and specific suggestions with respect to problems for algebra, geometry, and trigonometry. The range of topics and difficulty is pretty wide, as is the range of originality and appeal of the problems. The table of contents lists twenty-six problem categories; some have several problems included.

4. Beberman, Max, "Statement of the Problem," Proceedings of the UICSM Conference on the Role of Applications in a Secondary School Mathematics Curriculum, Feb. 14-19, 1963, Urbana, Illinois; 11.

Summary: Beberman undertakes to list arguments which show that you can't teach applications in junior and senior high schools! 1) Students have so little background in science. 2) Most mathematics teachers are not acquainted with applications. 3) Applications require elaborate equipment. 4) Young students, by and large, are not interested in the applications of the subject. 5) Significant applications may require more than one lesson to teach. 6) If you want to do it right, it may require a cluster of mathematical skills, only some of which the students may possess at the time the application is presented. 7) Applications tend to give students the idea that mathematics has no right to its own existence.

5. Buchalter, V. E., "The Logic of Nonsense," M. T., (May, 1962), 55: 330-333.

Summary: The discussion here revolves around the tales of Lewis Carroll. The examples from several of the Alice books are delightful, but the explanations of the logic and deliberate misuses of logic are less than adequate.

6. Davis, Philip J., "The Criterion Makers; Mathematics and Social Policy," American Scientist, (September, 1962), 258A-274A. (P. Peak, M. T., (January, 1963), 56:7.

Summary: The social scientists and the economists are no longer frightened of mathematics. They use it for determining proper job placement, marriage partners, college placements, flow of automotile traffic, small-town development and the best food for the least money. The mathematician revels in abstractions and generalizations; he changes hypotheses and looks for patterns of operation. Therefore, he is prepared to carry out the mathematical analysis, but the criterion is extramathematical.

7. Fehr, H., "The Role of Physics in the Teaching of Mathematics," M. T., (October, 1965), 56:394-399.

Summary: Reference is made to unpublished thesis (Douglas B. Williamson, "The Mathematics Essential to the Learning of Engineering Physics," unpublished D. Ed. report, TCCU, 1956). This study listed more than fifteen thousand instances in which mathematical concepts or processes from secondary school level mathematics were needed in the study of one text book. Fehr suggests that among these fifteen thousand instances there are surely a large supply of illustrations that could be used in the teaching of mathematics itself. He discusses specific ways in which physics can aid learning of mathematics via measurement, mathematical models, vectors, etc.

8. Fry, T. C., "Mathematics as a Profession Today in Industry," A. M. M., (February, 1956), 63:71-80.

Summary: In 1940 Fry arrived at the figure of only 150 mathematicians in industry and government combined. In 1955 in a conference on the training of applied mathematics, the need was estimated to be around 1,500. "The indispensable function of the mathematician is formulating problems, not solving them. This is even true of the mathematician who specializes in computation... As scientists from all parts of our organization bring problems to him for calculation, the first discussion almost invariably concerns the origin of the problem and its formulation; and it is not unusual for the scientist to go away convinced that he had brought a different problem than he had intended." Not many new ideas but useful to a student or a novice.

9. Gafney, L., "Gaspard Monge and Descriptive Geometry," M. T., (April, 1965), 58:338-344.

Summary: An historical article that has several interesting diagrams including the comparison of some of the diagrams on Monge's book with a painting by Durer. An extensive bibliography gives the basis for a further study of the subject. While not on applications as such, this is of interest for its linking of geometry to art and engineering.

10. Greenspan, H. T., "Applied Mathematics as a Science," A. M. M., (November, 1961), 68:872-880

Summary: A general introduction is followed by discussion of specific examples. First is a discussion of boundary layer theory. He mentions in passing plasma dynamics, gas dynamics, electromagnetic theory, astrophysics, materiology, oceanography, information theory, economics. He treats in somewhat more detail a problem of analyzing ocean waves stirred up by hurricanes and a problem in automobile traffic flow. No example is described in very much detail. A useful article for a teacher to get some idea as to what is meant by applied mathematics and what is going on in applied mathematics. Marginally useful for students, say, at the middle secondary level.

11. Gulliksen, Harold, "Mathematical Solutions for Psychological Problems," American Scientist, (June, 1959), 47:178-201.

Summary: In addition to several examples of the applications of matrix theory to psychological scaling, the author provides a good list of references and offers an additional list of one hundred articles on request. His address is Department of Psychology, Princeton University, Princeton, New Jersey.

12. Hart, R. W., and Wood, W. H., "Ratings of College Mathematics Courses by Applied Mathematicians," A. M. M., (June-July, 1959), 66:510-512.

Summary: This reports the results of a questionnaire sent to large concerns that hire mathematicians asking them to rate various college mathematics courses according to their desirability for their prospective employees. Tables list those courses that were rated as "most desirable" by more than 40 per cent of the mathematicians and the courses that were marked as "not needed" by more than 30 per cent of the mathematicians.

13. Jones, P. S., "Mathematical Demonstrations and Exhibits" in Multisensory Aids in the Teaching of Mathematics, Eighteenth Yearbook of the NCTM, Columbia University, New York, 1945, 88-103.

Summary: There are a number of interesting items in this article that could be adapted to illustrate applications, especially in the "Suggested Topics for Exhibits," p. 97-99. Of particular interest is a long bibliography that includes a number of pre-1942 articles containing applications and models.

14. Lyda, Wesley J., and Church, Ruby S., "Direct, Practical Arithmetical Experiences and Success in Solving Realistic Verbal 'Reasoning' Problems in Arithmetic," Journal of Educational Research, (July-August, 1964), 530-533. (P. Peak, M. T., January, 1965, 58:31)

Summary: "You will be interested to note that the above-average ability group can do problems where they have had no experience better than the below-average group; that direct experience is a valuable factor in problem-solving success of below-average students; that the experiences of the below-average group and the above-average group were different in both kind and amount; and that in some instances students had had no experiences relating to mathematical reasoning. This study was based on 175 pupils in Firt Valley, Georgia."

15. "Mathematics in Education and Industry," Supplement to the Mathematical Gazette, (December, 1963), 362:17. (Baum, J. D., A. M. M., November, 1965, 72:1018-19)

Summary: "This is a report of the experience of two teachers of mathematics who spent some time in industry (with British Petroleum) and of the conclusions they drew from that experience. The following are pointed out as being significant needs of people who must use mathematics in industry: knowledge of basic mathematics and the skill to apply these basic concepts in problems from a variety of sources; knowledge of what a computer can do, not how it does it; a knowledge of statistics, a knowledge of numerical methods; the ability to set up differential equations. An interesting point is made that the approach to abstract concepts is best made via the recognition of the concept as a common pattern in a variety of real situations."

16. Mosteller, F., "Contributions of Secondary School Mathematics to Social Science," Proceedings of the UICSM Conference on the Role of Applications in a Secondary School Mathematics Curriculum, Feb. 14-19, 1963, Urbana, Illinois; 1964, 85-109.

Summary: This is an excellent and comprehensive discussion with many nice examples given. He closes with these recommendations: "(1) Courses in probability and statistics; (2) Make available to guidance teachers more information about new [illegible] of mathematics, not only in the social sciences but also in [illegible]gy, medical research, and other lines of work; (3) Writing group be constructed to assemble a substantial body of problems appropriate to mathematics in the secondary schools; (4) Study the curriculum of secondary school mathematics just from the point of view of training for computation; (5) Pamphlets of the order of 25 to 100 pages on such topics as the inventory problem, elementary queuing theory and its applications, and the use of convexity in applied mathematics, say, economics; (6) Laboratory work is a neglected source of motivation; (7) There are many topics on which five-minute or ten-minute films would be a blessing and which I believe teachers would be delighted to use if they were available."

17. Northhover, F. H., "University Training in Applied Mathematics," Canadian Mathematical Bulletin, (July, 1964), 463-466. (P. Peak, M. T., May, 1965, 58:439)

Summary: "The author believes we have not met the need. The applied mathematician needs rigor but not complete generality. Today's courses offer only the latter, leaving no time for the student directed toward applied mathematics to learn the techniques of successful applications. The program today omits essential areas of study, such as methods of approximation, special functions, equations solved by special methods, and the opportunities to develop judgment and insights into applying methods to particular situations."

18. Rees, Mina, "Mathematicians in the Market Place, A. M. M., (May, 1958), 65:332-343.

Summary: This is a report on the *Survey of Research Potential and Training in the Mathematical Sciences* (University of Chicago, 1957). It shows that between seven and eight thousand men and women are thought of by their employers in nonacademic work as mathematicians, and nearly nine hundred Ph.D.'s are functioning as mathematicians though at most seven hundred of these had their original training in mathematics. She quotes from *Oxford Mathematical Conference* (abbreviated proceedings), London, 1957. "Only a handful of school teachers have ever used mathematics in practice... Solving equations is a minor technical matter compared with this fascinating and sophisticated craft of model-building, which calls for both clear, keen common sense and the highest qualities of artistic and creative imagination." She discusses what is expected of the Ph.D. mathematician in industry. She discusses some of the specific mathematical training needed for work in industry, and she suggests an internship arrangement to familiarize students with the industrial applications. Much of the article is now dated, but there is useful information in it. Her proposals seem pretty sound.

19. Rosentberg, H., "Modern Applications of Exponential and Logarithmic Functions," *School Science and Mathematics*, February, 1960, 131-138.

20. Schaaf, W. L., "Art and Mathematics: A Brief Guide to Source Materials," *A. M. M.*, (March, 1951), 58:167-177.

Summary: This is an extensive bibliography under the following categories: (1) General; (2) Ornament and Design; (3) Geometric Patterns; Repeating Designs; (4) Perspective; Painting; Geometric Drawing; (5) Architecture; (6) Dynanic Symmetry; (7) The Golden Section. The articles come from a number of sources including journals and books that one would not ordinarily think of as sources of mathematical material. Some of the titles have brief summaries. References go back to as early as 1842 and include some articles in French and German. The majority are pre-1940.

21. Stein, Sherman K., "The Mathematician as an Explorer," *Scientific American*, (May, 1961), 149-158. (P. Peak, *M. T.*, January, 1962, 55:51)

Summary: "This article gives us a different perspective. It gives that interplay between accumulated knowledge and the motivating forces of curiosity; it shows the relation of tastes with the needs of the times. This article is an excellent illustration of how the pure mathematics of one age becomes the applied mathematics of the next, how what to one age is useless but to another becomes vital."

22. Stone, Marshall, "The Revolution in Mathematics, A. M. M., (October, 1961), 68:715-734.

Summary: This article has been much quoted and is very long. A few scattered quotations may serve to indicate its content. "While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world... It may seem to be a stark paradox that, just when mathematics has been brought close to the ultimate abstractness, its applications have begun to multiply and proliferate in an extraordinary fashion... Far from being paradoxical, however, this conjunction of two apparently opposite trends in the development of mathematics may rightly be viewed as the sign to an essential truth about mathematics itself. For it is only to the extent that mathematics is freed from the bonds which have attached it in the past to particular aspects of reality that it can become the extremely flexible and powerful instrument we need to break paths into areas now beyond our ken..." Stone closes by summarizing developments in mathematical analysis, computation, logic, probability, and statistics, all in beautifully concise fashion. He talks at some length about the problems of mathematics education at the college level. All in all, it's a beautiful article that should be read by most teachers, I think, and by students of mathematics including school students. Reprints are available from the Conference Board of the Mathematical Sciences, Mills Building, 17 Pennsylvania Avenue, N. W., Washington 6, D. C.

23. The Teaching of Mathematics to Physicists, The Institute of Physics and Physical Society, London, 1960, excerpts reported in Mathematical Education Notes, A. M. M., (October, 1961), 58:798-801.

Summary: There are some useful observations in this paper. Included is a picturesque quotation from a Professor Synge: "The use of applied mathematics in its relation to a physical problem involves three stages: (1) a dive from the world of reality into the world of mathematics; (2) a swim in the world of mathematics; (3) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth." They draw a distinction between rigor, which must be maintained, and generality, which may often be sacrificed (for the physicist). They observe that for the physicist the distinction often drawn between the pure and applied mathematics becomes meaningless and may be misleading. They complain that applied mathematics in British universities often deals with problems that are not important or physically realizable but require ingenuity and have surprising solutions.

24. Tukey, J. W., "Mathematical Consultants, Computational Mathematics, and Mathematical Engineering," A. M. M., (October, 1955), 62:565-571.

Summary: Writing ten years ago, the author predicts the emergence of two sorts of mathematical specialists: 1) the consultant to industry and government, trained along the same lines as the research mathematician, 2) "mathematical engineering" (which he regards as more appropriate than "applied mathematics" as a title). He summarizes his point of view as follows: "If the moral drawn above is correct, then mathematics department should, in the writer's judgment, act as follows: 1) they should prepare to cooperate in the setting up of training for computation engineers under engineering auspices, and 2) they should prepare to teach the necessary mathematics including the mathematics of computation in the mathematics department."

25. Tukey, J. W., "The Teaching of Concrete Mathematics," A. M. M., (January, 1958), 65:1-9.

Summary: Although this article is directed at college teachers, the comments on computational methods, teaching a method of and the mathematics of computation, teaching intelligent use of tables, and the comments on "approximation" and "formulation" are certainly relevant to the teaching of applications in schools. His summary includes the following suggestions: 1) the teaching of any form of computation should be directed (in relatively elementary or general courses) toward the occasional computer rather than the steady computer; 2) this requires emphasis on simple, easy-to-learn-or-relearn methods; 3) a reduction in the labor of computation is the only visible way of finding the time and effort to make the study of computation more rewarding; (4) there is much to be done in connection with the development of applied mathematics of formulation and approximation.

26. Wigner, Eugene P., "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," Communications on Pure and Applied Mathematics, (February, 1960), 13:1-4.

Summary: This article is difficult to summarize, but I recommend it highly. Some random quotations follow: "...The first point is that mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections... What is Mathematics?... I would say that mathematics is the science of skillful operations with concepts and rules invented just for this purpose... Furthermore, whereas it is unquestionably true that the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play such an important role in physics... What is Physics? The physicist is interested in discovering the laws of inanimate nature... The world around us is of baffling complexity and the most obvious fact about it is that we cannot predict the future. It is as Schroedinger has remarked, a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered... The Role of Mathematics in Physical Theories...," etc.

27. Zassenhaus, Hans, "The Concept of Depth in Teacher Training," M. A. A., (January, 1965), 70:85-88.

Summary: The author talks mainly about the problem of retraining teachers in academic year Institutes and other programs. "My only criticism of the existing program of developing master teachers of mathematics lies in the objection that it does not include the applications to science in any way that is really meaningful. Because of a lack of understanding by my audience I have no chance to illustrate and indeed deepen the understanding of the concepts of vector, orientation, tensor, principal axis theory, centroid, even limit, in the way traditional in Europe... Suggest that A. Y. I. programs for mathematics be extended by a second year in which the applications of mathematics are emphasized."